AMERICAN
JOURNAL of PHYSICS

A Journal Devoted to the Instructional and Cultural Aspects of Physical Science

VOLUME 16, NUMBER 8 NOVEMBER, 1948

Energy Diagrams for Beta-Disintegration

A. L. HUGHES
Washington University, St. Louis, Missouri

THE results of investigations on the energies of beta- and gamma-rays from radioactive isotopes are frequently summarized in diagrams of the kind shown in Fig. 1. There is a lack of uniformity in the literature as to the labeling of the transitions. When a transition involves the emission of negative beta-rays (Fig. 1A) it is represented by a slanting line, usually marked with a number, such as 0.8 Mev, which measures the kinetic energy of the fastest beta-rays appearing in the beta-ray spectrum under consideration. However, it is occasionally labeled 0.8 Mev$+m_0c^2$, where the 0.8 Mev still measures the maximum kinetic energy of the beta-rays. When a positron transition is involved (Fig. 1B), one generally finds printed on the slanting line marked β^+, a quantity 0.3 Mev$+2m_0c^2$ where the 0.3 Mev is the energy of the fastest positrons. But one also finds in the literature the same line labeled with the quantity 0.3 Mev$+m_0c^2$ (that is, without the 2) where, as before, the 0.3 Mev is the energy of the fastest positrons. Some authors add a vertical scale in Mev units, but they seldom state explicitly what is measured in these units.

The purpose of this paper is to clarify the situation and to show how apparently different representations are reconcilable when properly interpreted.

The principal symbols that will be used are as follows:

Z: the atomic number of the isotope;
A: the mass number of the isotope;
N_Z: the exact mass of the *nucleus* of atomic number Z;
M_Z: the exact mass of the *neutral atom* of atomic number Z (M_Z and N_Z are in terms of 16 for oxygen);
m_0: the rest mass of the electron or positron;
β^-, β^+: negatively and positively charged beta-particles;
ν: the neutrino;
B_Z: the total binding energy of the electrons in atom Z.

The relation between the exact mass of an atom Z and that of its nucleus is given, to a high approximation, by

$$M_Z = N_Z + Zm_0. \tag{1}$$

If we take the binding energy into account, a more accurate equation is

$$M_Z = N_Z + Zm_0 - B_Z/c^2, \tag{2}$$

where B_Z/c^2 is the mass equivalent of the binding energy. This binding energy is the energy radiated when Z widely separated electrons come together around the nucleus to form the neutral atom. Since in most cases the binding energy comes into the final formula as a difference, $B_Z - B_{Z-1}$, and since this difference is much smaller than both the usual beta-ray kinetic energy and the energy equivalent of the mass of an electron, it will be neglected in Part I immediately following, but will be taken into account in Part II.

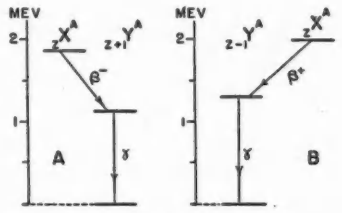


FIG. 1. A. Beta-decay (β^-) followed by a gamma-ray. B. Positron decay (β^+) followed by a gamma-ray.

I. Binding Energy Neglected

β^--Decay. A transmutation accompanied by the emission of negative beta-particles may be represented symbolically by

$$ {}_ZX^A \rightarrow {}_{Z+1}Y^A + \beta^- + \nu + T, \tag{3}$$

which states that the parent element X transmutes into a daughter element Y accompanied by the emission of a beta-particle, a neutrino and kinetic energy T. The kinetic energy is carried by the beta-particle and the neutrino and, to a very much smaller extent, by the recoil nucleus. The energy T is divided in all proportions between the beta-particles and the neutrinos. Experimentally one identifies T with the energy of the fastest beta-particle. We shall now write down the mass balance in terms of the exact masses of the nuclei (not the atoms) when a nucleus emits a negative beta-particle. (The neutrino has no mass.)

$$ N_Z = N_{Z+1} + m_0 + T/c^2. \tag{4}$$

Here T/c^2 is the mass equivalent of the kinetic energy carried by the beta-particle, the neutrino and the recoiling nucleus. We shall now use Eq. (1) to express this in terms of exact atomic masses (not nuclear masses).

$$ \begin{aligned} M_Z - Zm_0 &= M_{Z+1} - (Z+1)m_0 + m_0 + T/c^2, \\ M_Z &= M_{Z+1} + T/c^2. \end{aligned} \tag{5}$$

Equation (4) states that the parent nucleus must have enough mass to provide for the mass of the daughter nucleus, that of the beta-particle (m_0) and the mass equivalent of the kinetic energy T. Equation (5) states that the mass of the parent *atom* must provide only for the mass of the daughter *atom* and the mass equivalent of the kinetic energy T. The reason for the disappearance of m_0 from Eq. (5) is that, immediately after the emission of the beta-particle, the daughter atom still has Z atomic electrons but a nuclear charge $Z+1$, and so it is a positively charged ion. It becomes neutral by absorbing an electron from outside. Thus this incoming electron just compensates for the emitted beta-particle. Equations (4) and (5) are represented graphically in Fig. 2.

β^+-Decay. When the transmutation is accompanied by the emission of a positron (β^+), it may be represented symbolically by

$$ {}_ZX^A \rightarrow {}_{Z-1}Y^A + \beta^+ + \nu + T. \tag{6}$$

As before, the mass balance in terms of exact nuclear masses (not atomic masses) is given by

$$ N_Z = N_{Z-1} + m_0 + T/c^2. \tag{7}$$

To find the mass balance in terms of exact atomic masses we substitute Eq. (1) in Eq. (7) and get

$$ \begin{aligned} M_Z - Zm_0 &= M_{Z-1} - (Z-1)m_0 + m_0 + T/c^2, \\ M_Z &= M_{Z+1} + 2m_0 + T/c^2. \end{aligned} \tag{8}$$

Equations (7) and (8) are represented graphically in Fig. 3.

It is to be noted that Eq. (7) is symmetrical with Eq. (4), while Eq. (8) differs from Eq. (5) in the appearance of a $2m_0$ in the former. This follows from the fact that when a positron leaves the nucleus of the atom, the atom must also lose an electron for the daughter atom to become neutral. Hence two particles, a positron and an electron, leave the *atom* and so we have the $2m_0$, whereas in the β^+-decay of a bare nucleus only the positron leaves it.

Some writers state that in positron emission the process has to provide energy to create a positron-electron pair. This happens to be true numerically (Eq. (8)) but it carries with it an

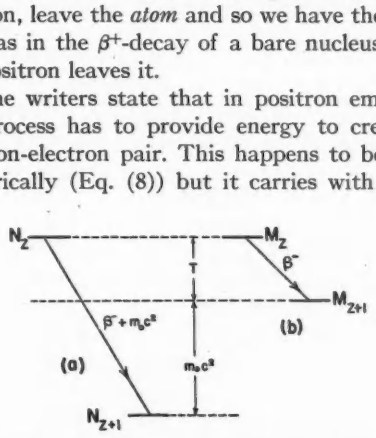


FIG. 2. Representation of a β^--transition in terms of (a) exact nuclear masses, (b) exact atomic masses. T = kinetic energy released; m_0c^2 = energy equivalent of the mass of the β^--particle.

unwarranted implication. There is no pair production in the sense that the positron and electron are created simultaneously as when, say, a photon disappears. It is sometimes implied that β^+-decay is fundamentally different from β^--decay in that in the former positron-electron pairs are created. The situation is due to the fact that mass spectroscopists invariably list the exact masses of the neutral atoms but not those of the nuclei. Consequently in order to make use of the exact *atomic* masses, as tabulated, one must make use of the asymmetrical Eqs. (5) and (8), one containing a $2m_0$ and the other not. Had it been the practice to tabulate exact nuclear masses, which in some respects are more fundamental in nuclear physics, the beta-ray spectroscopists would have used the perfectly symmetrical Eqs. (4) and (7), and the rather extraneous consideration of pair creation would not have intruded itself. If it were possible to study decay of bare radioactive nuclei, only Eqs. (4) and (7) would apply.

Decay by *K* Capture. A process which sometimes competes with β^+-decay and sometimes replaces it altogether is known as K capture. The nucleus captures an electron from the K level of the atom, and as a result its atomic number changes from Z to $Z-1$ just as in the case of β^+-decay. (Electrons can be captured, though usually to a much less extent, from the L and M shells also. The argument is the same in all cases.) A K capture decay may be represented symbolically by

$$ {}_ZX^A + e_K \rightarrow {}_{Z-1}Y^A + \nu + T. \qquad (9) $$

Here the kinetic energy T is that of the neutrino. The neutrinos will be monoenergetic since there

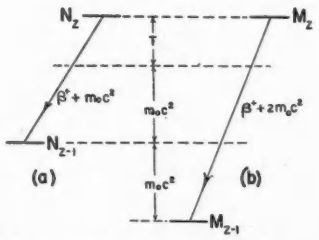


FIG. 3. Representation of a β^+-transmutation in terms of (a) exact nuclear masses, (b) exact atomic masses. T = kinetic energy released; m_0c^2 = energy equivalent of the mass of the electron or positron.

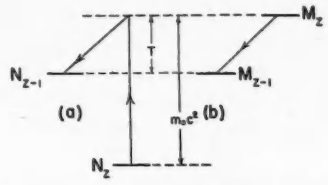


FIG. 4. K capture transmutation in terms of (a) exact nuclear masses, (b) exact atomic masses. T = kinetic energy of the neutrino; m_0c^2 = energy equivalent of the mass of the electron captured.

are no positrons to share the energy with in varying proportions. In K capture it is essential that there be an atom to provide the K electron, and so it is appropriate to start with the equation for the energy balance in terms of exact atomic masses

$$ M_Z = M_{Z-1} + T/c^2. \qquad (10) $$

This states that the parent atom has to provide for the mass of the daughter atom plus the mass equivalent of the kinetic energy of the neutrino. Equations (1) and (10) yield

$$ N_Z + m_0 = N_{Z-1} + T/c^2. \qquad (11) $$

While Eqs. (4) and (7) describe situations which are readily pictured, the radioactive decay of isolated nuclei by K capture has no meaning for an isolated nucleus. To give it meaning one has to provide a single electron to be captured by the isolated nucleus. Equations (10) and (11) are shown in Fig. 4.

It is instructive to consider cases in which both β^+-decay and decay by K capture occur as competing processes, and in which only K capture can occur. These are shown diagrammatically in Figs. 5 and 6. It is to be noted that when competing processes occur the energy carried away from the atom by the neutrino in K capture decay is always greater by $2m_0c^2$ than the energy carried by the positron in the alternative β^+-decay. However, it is not possible to measure directly the energy carried away by the neutrino when K capture occurs. Indirectly one can attempt to measure it by determining the recoil energy of the nucleus, which is very difficult to do experimentally.

Energy Assignment to Nuclei. In the first paragraph of this paper, mention was made of the practice of some authors of laying off a vertical

scale in Mev units alongside a transmutation diagram. In general there is no explicit statement as to whether such a scale refers to neutral atoms or to nuclei. This can perhaps best be discussed by considering a specific case, that of the decay

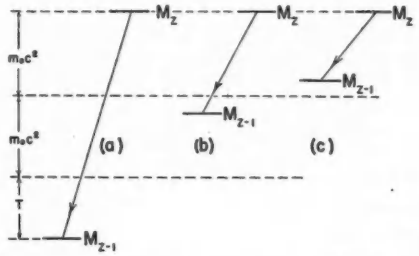


FIG. 5. K capture and β^+-decay. (a) Both processes possible, the kinetic energy of the positron and its associated neutrino equals T, and that of the neutrino in K capture being $T+2m_0c^2$. (b) and (c) K capture possible, β^+ emission not possible.

of Cu^{64} by β^--emission to Zn^{64} and by β^+-emission or K capture to Ni^{64}. Figure 7(a) is a copy with slight modifications of the decay scheme for Cu^{64} as published by Bradt and others.[1]

Although this was not mentioned explicitly by the authors, it is clear that the vertical scale refers to atomic energies and not to nuclear energies, for the following reasons. (1) The energy difference between Cu^{64} and its daughter Ni^{64} by β^+-decay is $2m_0c^2$ greater than the β^+_{max}, the kinetic energy of the fastest positrons. (2) The energy difference between Cu^{64} and its daughter Zn^{64} is the same as the energy value for β^-_{max}. These fit in with Eqs. (8) and (5). The corresponding diagram in terms of nuclear energies is shown in Fig. 7b. Here the differences in energy values between the parent nucleus, Cu^{64}, and each daughter nucleus, Ni^{64} and Zn^{64}, contain, in each case, the energy m_0c^2 added to the kinetic energies β^+_{max} and β^-_{max}, respectively, in accord with Eqs. (4) and (7).

Gamma-Ray Assignments. It is worth while calling attention to a difference in practice between certain authors in the way they associate gamma-rays with elements. The majority list a gamma-ray following a beta-ray as belonging to the same element. Thus, referring to Fig. 7 the positrons, the (negative) beta-rays, the K capture and the gamma-rays are generally lumped together as radiations associated with Cu^{64}. It is more logical, as some authors do, though perhaps less convenient, to state that the 1.20-Mev gamma-ray belongs to Ni^{64} and is given out in a transition from an excited state of that isotope to the ground state.

[1] Bradt *et al. Helvetica Physica Acta* 19, 220 (1946).

II. Binding Energy Included

β^--Decay. Consider a system consisting of a neutral atom of atomic number Z and an isolated electron at rest. The total mass is M_Z+m_0. When beta-decay occurs the nucleus changes in atomic number to $Z+1$ and the isolated electron falls into the atom to give a neutral atom whose mass will be M_{Z+1}. The beta-particle will carry away mass m_0, and kinetic energy T shared by the beta-particle and the neutrino, will appear. The exact atomic mass balance is

$$M_Z+m_0=M_{Z+1}+m_0+T/c^2,$$

or

$$M_Z=M_{Z+1}+T/c^2. \tag{12}$$

This is identical with Eq. (5) and the same remarks apply as were made about Eq. (5). Let us convert this to exact nuclear masses using Eq. (2) instead of Eq. (1). We get

$$N_Z=N_{Z+1}+m_0+T/c^2-(B_{Z+1}-B_Z)/c^2 \tag{13}$$

which differs from Eq. (4) by a small quantity involving the difference of the binding energies.

It is instructive to consider the case of beta-decay of a bare nucleus, to which

$$N_Z=N_{Z+1}+m_0+T'/c^2 \tag{14}$$

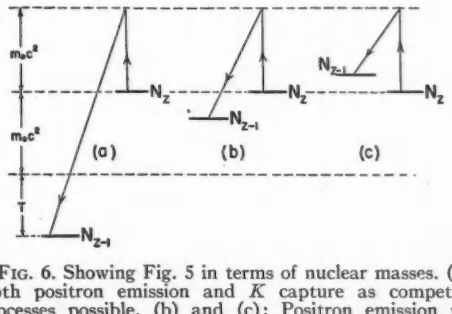


FIG. 6. Showing Fig. 5 in terms of nuclear masses. (a): Both positron emission and K capture as competing processes possible. (b) and (c): Positron emission not possible; K capture possible. (c) shows that K capture can occur even if daughter nucleus has more mass than parent nucleus provided the difference does not exceed m_0c^2.

applies. We see then that the kinetic energy appearing in the β^--decay of a bare nucleus differs slightly from that in the β^--decay of a neutral atom. The relation is

$$T' = T - (B_{Z+1} - B_Z). \tag{15}$$

This means that when the electronic structure changes from that characteristic of atom Z to that characteristic of atom $Z+1$, energy is made available by the amount $(B_{Z+1} - B_Z)$ and so T exceeds T' by just this amount.

β^+-Decay. In this case we have initially a single neutral atom of mass M_Z. When a positron is emitted from the nucleus and followed immediately by the emission of an electron from the electronic structure to give a neutral atom of mass M_{Z-1}, the mass balance is given by

$$M_Z = M_{Z-1} + 2m_0 + T/c^2. \tag{16}$$

This is identical with Eq. (8). On combining Eq. (2) with Eq. (8) we obtain

$$N_Z = N_{Z-1} + m_0 + T/c^2 + (B_Z - B_{Z-1})/c^2 \tag{17}$$

which differs from Eq. (7). Consider what happens when the bare nucleus of the same atom undergoes positron decay. The exact nuclear mass balance now is

$$N_Z = N_{Z-1} + m_0 + T'/c^2. \tag{18}$$

Comparing this with Eq. (15) we find

$$T' = T + (B_Z - B_{Z-1}). \tag{19}$$

This means that since energy $(B_Z - B_{Z-1})$ has to be supplied to change the electronic structure characteristic of atom Z to that of atom $Z-1$, T is *less* than T' by this amount. Equations (15) and (19) are unsymmetrical with respect to the sign of the second quantity on the right. In the β^--case, the rearrangement of the electronic structure from that characteristic of a parent atom to that of a daughter atom containing one more electron sets free a small amount of energy. In the β^+-case, the rearrangement from that characteristic of a parent atom to that of a daughter atom containing one less electron demands a small amount of energy to put it through.

***K* Capture.** When the nucleus of the atom Z whose exact mass is M_Z captures a K electron only a neutrino leaves the atom and it carries with it the energy T. The capture of a K electron leaves a hole in the K shell and as this is filled by rearrangement of atomic electrons, x-rays are emitted. Let $B_{Z-1}{}^K$ denote their

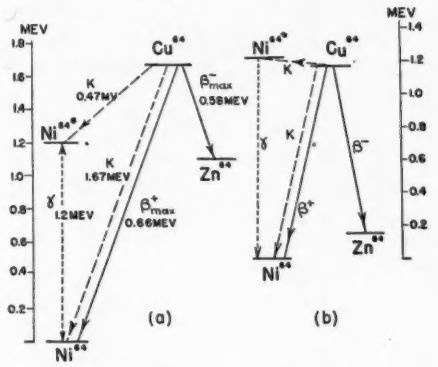


FIG. 7. (a) Disintegration scheme of Cu^{64} in terms of exact atomic energies as given by Bradt *et al.* (b) The same in terms of exact nuclear energies.

energy. We then have

$$M_Z = M_{Z-1} + T/c^2 + B_{Z-1}{}^K/c^2. \tag{20}$$

On combining this with Eq. (2) we have

$$N_Z + m_0 = N_{Z-1} + T/c^2 + B_{Z-1}{}^K/c^2 + (B_Z - B_{Z-1})/c^2.$$

Whereas it is possible to imagine both β^- and β^+-decay for a bare nucleus, as we have done to obtain Eqs. (14) and (18), there is no analog for K capture, unless we consider the rather artificial situation of a bare nucleus capturing an isolated electron.

The consideration of binding energies was brought in to make the discussion more complete. They enter into some equations as a difference $(B_Z - B_{Z-1})$ which is almost negligible in comparison with the kinetic energies measured in beta-decay experiments. In any case, if one uses exact atomic masses, as one does in Eqs. (12), (16) and (20), the difference $(B_Z - B_{Z-1})$ does not appear. It would appear, if one used exact nuclear masses. As an illustration of the magnitudes involved, $B_Z - B_{Z-1}$ for $Z = 82$ is approximately 0.015 Mev. In the case of K capture, the binding energy of the K electron enters into the exact atomic mass equation, Eq. (20). This

carries with it the implication that if M_Z exceeded M_{Z-1} by an amount insufficient to provide for $B_{Z-1}{}^K/c^2$, then K capture could not occur although L capture would be possible energetically if $M_Z - M_{Z-1}$ exceeded $B_{Z-1}{}^L/c^2$. To illustrate this numerically we note that B^K and B^L are, respectively, 0.088 Mev and 0.016 Mev for lead. For a radioactive isotope of bismuth to decay into lead by K capture the energy equivalent of the exact atomic mass of bismuth would have to exceed that of lead by more than 0.088 Mev for K capture, but only by 0.016 Mev for L capture.

The author wishes to express his appreciation to Dr. H. Primakoff for his helpful comments and suggestions. This article was written as part of a program of research supported by the Office of Naval Research.

New Members of the Association

The following persons have been made *members* or *junior members* (*J*) of the American Association of Physics Teachers since the publication of the preceding list [*Am. J. Physics* **16**, 414 (1948)].

Barker, William Alfred II (J), 252 Arbolada Dr., Arcadia, Calif.
Furnas, Thomas C., 4826 Whitewood Ave., Lakewood Village, Long Beach 8, Calif.
Greene, Lawrence C. (J), 116 E. Hammond St., Otsego, Mich.
Greenspan, Lewis, 353 Division St., Amsterdam, N. Y.
Homan, John G., Box 350, Steubenville, Ohio.
Lee, Iva Helen, 427 Vine St., Colorado City, Tex.
Lemmerman, Carl Walter, 45 Granby St., Hartford, Conn.
Levinger, Joseph S., 111 W. York St., Ithaca, N. Y.
Levinson, Irving J., 6519 London Ave., Detroit 21, Mich.
Luke, Perry James, Sloane Physics Laboratory, Yale University, New Haven, Conn.
Marchesi, George Joseph, 102 Alkamont Ave., Scarsdale, N. Y.
Maurer, Dr. Robert Joseph, Carnegie Institute of Technology, Pittsburgh, Pa.
McGaughey, John R., 3607 Talbot St., San Diego 6, Calif.
Miller, Macon Hendricks, 2814 Brooklyn 1, Detroit 1, Mich.
Mullikin, Forrest T., Transylvania College, Lexington, Ky.
Murphey, Byron F., 1115 13th Ave. SE, Minneapolis 14, Minn.
Parker, Floyd W., Lincoln Memorial University, Harrogate, Tenn.
Petersen, Fred L. (J), 506 Lane H., Hastings, Nebr.
Ramberg, Edward G., Bryn Gweled, Feasterville, Pa.
Rider, John Francis, 404 Fourth Ave., New York 16, N. Y.
Roys, Paul Allen (J), 115 W. Park Blvd., Villa Park, Ill.
Schultz, Frederick H. C., Box 303, University Station, Grand Forks, N. D.
Shetler, W. Lewis, Lt., U. S. Maritime Service, Department of Engineering, U. S. Merchant Marine Academy, Kings Point, N. Y.
Slepian, Joseph, Westinghouse Research Laboratories, E. Pittsburgh, Pa.
Smith, Paul L., Crystal Section, Naval Research Laboratory, Washington 20, D. C.
Stewart, Peter A. R., c/o Physics Department, University of Minnesota, Minneapolis, Minn.
Venable, William Mayo, 6111 Fifth Ave., Pittsburgh 6, Pa.
Weisman, William, c/o Physics Department, Long Island University, Brooklyn 1, N. Y.
Welch, M. W., Welch Scientific Company, 1515 Sedgwick St., Chicago 10, Ill.
Wightman, E. Russell, Houghton College, Houghton, N. Y.
Williams, Marvin M. D., Mayo Clinic, Rochester, Minn.
Winter, Wesley W. (J), Sterling Hall, University of Wisconsin, Madison, Wis.

Erratum. In the article, **Airy's Theorem and the Improvement of Clocks,** by PAUL F. GAEHR, *Am. J. Physics* **16**, 336–344 (1948), the expression $[OA_2 = OA_2]$ on p. 341 should be $[OA_2 = OA_1]$.

The Ph.D. Thesis and the College Teacher

G. W. Stewart
State University of Iowa, Iowa City, Iowa

FOR twenty years the graduate schools of the country have been urged to improve the training of college teachers by familiarizing these students with education itself, and by reducing over-specialization. But those in charge of awarding graduate degrees in the universities have not been satisfied with suggested alterations. Research scholars have been very certain that the primary purpose of the graduate work of the prospective teacher is to give him the best possible understanding of his field and to supply an experience in serious research requiring originality and contributing to knowledge.

The situation has recently been altered by commendable changes in the undergraduate college. Emphasis is now being laid upon a "general education" which will better prepare the student for the common life among us. At the same time the curriculum of the undergraduate has been broadened. This general education movement has already demonstrated its worth and is rapidly becoming a permanent asset of liberal education. A new emphasis has been placed upon methods of teaching for two reasons: first, much new planning has to be done in developing the teaching materials needed for these courses; second, the general education courses are more difficult to teach than the purely subject-matter ones and require new procedures. For the primary aim of the former is not to impart a body of facts. Rather it is to develop those skills which will better enable students to understand problems confronting them in all areas of activity in their common life. This aim is difficult to achieve. Each field is peculiarly qualified to make this or that contribution. Thus, with each subject, there has recently been given a careful consideration of especially appropriate methods of teaching. Experience in applying them has accumulated. So the prospective teacher may now find in his own department of specialization those who will gladly give him information about and instruction in teaching methods. Moreover, there is reason to expect that a plan of internship will soon be found in many university departments. An emphasis upon teaching methods will then have become the department's business and this policy will be welcomed instead of opposed by the creative scholars of the faculty.

Such changes will indirectly alter one of the criticisms that have been leveled at the graduate training of college teachers, namely, the lack of sufficient knowledge of teaching. It is advisable to have in mind, as the discussion proceeds, other opinions which have been expressed. One is to the effect that so much time is being spent on the research[1] thesis required for the Ph.D. degree that too little is left for obtaining a *broad* grasp of the specialist's subject. The thesis, it is said, results in narrow over-specialization. Again, the Report of the President's Commision on Higher Education[2] objects to a "single-minded emphasis on the research tradition" of the graduate schools as not conducive to the production of the kind of college teachers needed. The Report does not favor the weakening of the research function of the graduate school, but its strengthening. The Commission believes[3] that the opportunity for training in the principles of research "should continue to be universally available to all who are preparing for a career in higher education," and also that, "the faculty member who is to grow professionally and preserve his vitality of outlook must be equipped to work independently."

Evidently a careful consideration of the educational value of the research thesis to the prospective teacher can neither be shelved nor wisely delayed. The present discussion is therefore a timely one. We can proceed more definitely if all kinds of research theses and all types of graduate students are not included in the discussion. The

[1] The suggestion is made that the reader should not limit the word "research" only to that technical type so common in science, but that he include all manner of theses which are the seriously creative kind. These could contain creative criticism, creative interpretation and creative production in all fields in liberal arts. The discussion of this paper has a definite measure of application in any field.

[2] Vol. I, p. 89, U. S. Gov. Printing Office, 1947, now obtainable in a single volume under the title *Higher education for American democracy.*

[3] Vol. IV, pp. 19 and 20, U. S. Gov. Printing Office, 1947.

assumption will be made that the thesis, although for the purpose of a contribution to knowledge, is administered definitely for the student's benefit in that it enables him to accomplish something creative. Also it is assumed that the student has the ability to complete the thesis in a satisfactory manner and that he is adapted to the profession of teaching. The inquiry concerns the possible benefits of this creative experience to the student as he strives to make a contribution to knowledge. Why are the results of this experience a vital and essential part of his training for college teaching? The possible educational values of the thesis are chiefly the following:

1. It is generally admitted that in such a student the thesis experience will develop a craving for creative activity. True, he may not be able to carry out the investigations in which he is most interested at the time of accepting a teaching position. But the desire is there and his happiness will not be complete unless a part of his time is spent in productive activity in one form or another. Generally, but not always, the result will be something which is creatively satisfying. Publication should not be used as a measure of its value. The lively, growing quality of the teacher's mind will be an inspiration to his students.

2. Such an experience, if successful, will develop a new quality of self-respect and a greatly increased self-confidence in his creativeness. Why is this new to him? Because a graduate student's education has previously been quite free from any serious creative effort. In college he has been given some assignments which actually involved productive thinking, but of this as a purpose the student has not been made conscious. College education, by and large, has as its goal factual information; its lectures, its assignments and its examinations are so pointed. The teacher may be pleased with any original thinking of the student, but the methods of teaching, the plan of the course, and the crowning achievements as represented in the examinations stress factual information or knowledge.

3. This creative experience gives a sharp edge to one's ability to distinguish in another a creatively productive mind. This is a valuable asset, for all through life the individual will, if successful, pass judgment upon the potentialities of those around him, be they colleagues or beginners.

4. This experience can develop new habits of thought. For example, one may learn to cultivate ideas until each either proves itself worth while, or, as usually happens, must be cast aside. This habit is possible only with a person having faith in the result of cultivating the new ideas which pop into his mind, all the way up to the point where each will give a definite indication of its promise. This requires patience. One must be willing to pay the price of this labor of cultivation. The necessary confidence is developed in a successful creative experience.

5. In many kinds of creative effort one is taught with unmistakable emphasis that factual information is a means in the growth of the mental life and not its high goal. Though this distinction is highly important, it is hard to maintain in an environment where the machinery of operation (as in education, generally) tries to obliterate all else but factual information.

6. Concomitantly with the aspect just mentioned, a creative experience shows that the ability of recalling what has been memorized is only a handmaiden to the mind. This has significance in any vocation.

7. A serious effort in productive thinking or creativeness cultivates the habit of recognizing new problems that were formerly passed by without notice. Only a little imagination enables one to see what a profound influence this changed attitude of mind may have on the success of the individual in his chosen field.

8. The creative experience in the Ph.D. thesis enhances one's interest in the problems of fields other than his own. These fields no longer appear as bodies of static knowledge This broadened view has a definite significance in one's estimate of the relative importance of the problems in his own field from which his choice must be made.

By a rapid review one can see that creative effort is peculiarly well qualified to produce the educational benefits which have just been cited. It is noticed that the craving to become productive is definitely new, and that the quality of self-confidence is unusual, for it is a confidence in one's ability to do productive or creative thinking. Again, he who learns best to recognize creative ability is one who has passed through the struggles of original thinking of a productive character. Only he who has cultivated unpromising ideas and brought some of them to fruition can have the faith requisite to the thoughtful harboring of ideas which flit through the mind. What experience leads to an habitual search for new problems or new questions concerning the unknown? Clearly, only he who has learned the necessity of seeking new problems and the rewards which eventually come therefrom, will encourage this habit. Certainly this constant awareness of new problems lurking around the corner causes one to become more interested in current problems in all fields. On the other hand, the years spent by the student in gaining factual information, and understanding what others have said and written, builds up a great respect for accumulated knowledge. Written examinations taken in course have rewarded the ability to recall rather than to think on a higher plane. Only a serious creative effort subsequent to all this training can cause the student to become acutely aware of the value of the higher processes

of the mind. He still regards learning in the ordinary sense as a necessity but no longer as a worthy goal. All of these educational benefits accruing from a serious effort in productive thinking would seem difficult to acquire in any other way. To gain these values, the student must change an attitude acquired in a long educational experience.

Emphasized further, the fact must be realized that students are often told in class discussion that the suggestion just made by one of their number had been thought of a long time ago. The advice implied is "don't try to think of something new." The teacher does not have prominently in mind that the thinking process of the student in such a case may be just as productive or creative as if it had actually occurred for the first time in history. *Productive or creative thinking* is, in fact, not a distinct goal in college or even in graduate training, save the one concentrated experience in the preparation of the Ph.D. thesis. Teachers generally will question the truth of this statement, but the students themselves will attest to its accuracy. They will say, as undergraduates have in a special inquiry of the writer, that though they have been given what they now recognize as creative assignments, yet these were not specified as such at the time. Seldom, if ever, they will say, has creativeness been stressed either publicly or privately. It is *conscious* creative effort about which we speak. For it is the conscious kind which will increase one's self-confidence in his ability in productive thinking. The Ph.D. thesis, no matter how humble, alters the attitude, for the mind thereby has acquired a pride not in what it has stored, but in what it can and may do. This is a pride which should forever urge one to self-expression, to further contributions to his field of interest. Even though conditions may become adverse and opportunities meager, the mind is, because of its creative experience, much more likely to be continually creative, even though along a different line from that of the thesis.

What have the values herein stressed to do with college teaching? The question can be illustrated by the case in physics. We teachers know that physics is required in the engineering course for the purpose of instilling the principles of the subject and showing how to apply them in simple problems. Premedics require also a similar training, though in milder form. The purpose of physics in the curriculum is thus "to teach the student physics." Now comes the new and important interest in general education, which is definitely intended to be a training for the common life. Such an objective gives another purpose to a general education course in physics. It is no longer a pre-preparation for a profession but an education for the common life of us all. What can physics contribute to the mental habits, the insights, the outlooks, the manner of raising questions and stating problems, the confidence in one's ability to think independently, the awareness of the physical world and its significance? All these and other purposes of physics in general education come at once to the teacher's mind. But how can the teacher encourage independent thinking if he has not had a serious struggle himself? How can he be sensitive to the creative opportunity of each and every one in daily life, unless he has had a creative struggle? Who can best prepare a student for living, one who is full of knowledge or one who has had his own mind awakened to living at its best? Such questions lead definitely to the need for a thorough-going creative experience in the training of a college teacher. No amount of knowledge, no amount of broad learning can possibly be a substitute for this unique experience of the graduate student.

To be sure, a successful thesis of the creative type requires a considerable fraction of the time devoted to graduate study. Its removal would increase opportunity for the acquisition of factual knowledge in a graduate curriculum. But, as everyone must realize, the college teacher will, throughout life, continue to add to his knowledge. This is a fixed habit with him and the amount he can acquire in his graduate years is but a small fraction of what he will eventually master. Moreover, his basic broadening belongs chiefly in his undergraduate experience. Certainly he should have a broad basis. One is reminded of a mechanical, but nevertheless clarifying, analogy. A razor carries a sharp edge. But this must be prepared on a relatively broad

base in order to secure ample strength and support. The broad base exists only to carry the sharp edge. No one should think of establishing a base and neglecting the edge.

The idea that training for subsequent researches must be the chief object of a Ph.D. thesis in physics is unsound. The thesis has a much broader purpose, especially for the prospective teacher. Indeed, the new desire acquired for creative effort coupled with the broad benefits already mentioned produce a new attitude; and all performance depends importantly upon attitude. An examination of biographical accounts of research men in *American Men of Science* shows how very few remain in a narrow field where no new skills need to be acquired. Actual experience is a better preparation for any research of the future, wholly aside from the special skills developed in the thesis effort. In physics so rapid are the changes in methods and instruments that the skills mastered as a graduate student are only a fraction of what is subsequently necessary in the same narrow field.

Some physicists, who had theses subjects in what may now be described as classical physics, may question the future college teacher's advantage in a close association with a mighty cyclotron. For how far removed from simple college physics are such modern implements of research! But a more careful consideration will show that the future college teacher should have a good understanding of modern concepts and of research problems pertaining thereto. As our adjustment to a rapidly growing science continues, it will become more clear that differences in equipment are of secondary importance. They in no wise diminish the values of the research as a preparation for the college teacher. He must assist his students in their understanding of modern concepts and he must help them to peer into the future development of these ideas. It is the teacher's attitude, his habits and his interest which will greatly aid.

The imitative in men is very strong. The graduate student in physics can easily acquire the impression that there is only one kind of creative work deserving of recognition as such. It is that technical type represented by theses in experimental and theoretical physics in graduate departments of physics. Yet the college teacher of physics is surrounded by many other opportunities for productive activity. Whatever he can do in his lecture or classroom that involves something novel to himself is creative and should be so regarded. His students can be constantly impressed with the attitude of independent thought which results in something new. If one of the high purposes of general education is to cultivate the everyday creativeness of the student, then the teacher must live that way himself. The continuance of an elaborate investigation like that of his thesis is splendid if there is such an opportunity. But keeping creativeness alive does not depend upon that possibility. Neither, as already stated, is the publication of any minor creative thing he does the only evidence of his contribution. The sum of such frequent but individually small efforts, may, over the years, be accumulatively large in effect.

This paper calls attention to the importance of the contribution made to a college teacher's future by an adequate experience in a research thesis. It is implied that those directing these theses will be sensitive to the benefits of such an experience and make them as large as may be possible. Most of the "learning" which alters our habits and attitudes and thus our futures is commonly recognized as having been obtained only at the cost of experience. Man is creative by nature, but after many years devoted to factual learning he surely needs an experience to bring to active life this dormant ability.

No university is worthy of the name, that does not do everything in its power to promote original research in its laboratories. It is the duty of the university to see that its professors and teachers are not overburdened with routine teaching, but are given time for investigation and provided with research laboratory facilities and the necessary funds for this purpose.—E. RUTHERFORD

The Responsibility of the Teacher of College Physics for the Student's Facility in American Prose

PEARL I. YOUNG
Pottsville Undergraduate Center, The Pennsylvania State College, Pottsville, Pennsylvania

THE interrelation between a scientific education and a facility in English prose has been pointed out during the past twenty years in numerous articles in professional magazines devoted to various phases of engineering education. English teaching itself has been criticized on the ground that the material taught as "English composition" is not functional. The trend is definitely in the practical direction: to teach American rather than English prose, and to teach effective communication rather than rules of correct grammar.

The present trend is toward general education and the teaching of survey courses in physics and other sciences to the student in the liberal arts curriculum. This trend is going to force the science teacher into a closer understanding of the material being taught in the nonscientific courses. No educator would question the thesis that the physics teacher should present in approved present-day form and terminology the related material of mathematics, engineering drawing, and chemistry. He is now forced to learn some of the terminology and philosophy of the social scientist. It is high time that he took cognizance of the material stressed in courses in effective American prose.

Let it be very plain that nothing in this discussion is to be taken to indicate that the primary duty of a physics teacher is not to teach the principles of physics—correctly and completely. But while a physics teacher is fulfilling his primary duty of teaching physics, cannot he cooperate with the English teacher and teach it grammatically?

Although addressed, for simplicity, to teachers of college physics, all the following material applies equally well to teachers of college mathematics, chemistry, astronomy, geology, and engineering.

The Student's Need for Training in Technical Prose

No teacher need be reminded that, if the student is to be adequately prepared for earning his living and for living a full life, he should know what constitutes correct composition and should have acquired a facility in expressing effectively his thoughts on both technical and nontechnical subjects. He looks for guidance and example to his teachers of science and engineering. Everything he hears and reads helps to mold his forms of expression—so significant to his later advancement.

When the engineering student reaches college, he settles down seriously to prepare himself for his greatest usefulness to society. Thousands of these students enter college each year: keen, serious, eager to learn, malleable. Surely they have a right to expect of their college teachers, if not perfection, the very best they can offer. The student has set apart four or five years for the business of learning all the material that will help him in his profession and as a citizen. He has a right to expect from his teachers in the basic professional fields some consistency in facts, outlook, and emphasis; complete formal training in strictly undergraduate fields; the last word in modern information in these fields; a receptive attitude toward changing concepts; and a knowledge of authoritative sources from which to keep up to date.

The student realizes that if he is to progress after graduation he will have to continue his study of industrial applications in his chosen field, and of professional advances as presented in magazines and new textbooks. He hopes that he has received final preparation in other fields.

The Teacher's Responsibility for Effective Prose

For fifteen years as a technical editor in a research laboratory, I worked with incoming graduates of scores of engineering and technical schools. They had been awarded degrees of Bachelor of Science, Master of Science, and Doctor's degrees in Science, Engineering, or Philosophy. I have heard their remarks and noted their reactions as they were told that their command of technical

writing, or of everyday writing and speaking, was inadequate for advancement in their jobs. Their attitudes were abject, apologetic, scornful, or resentful.

Has such a graduate a right to be resentful? He writes and talks in the same style that passed in his mathematics, physics, and engineering classes. He copies the form, the words, and the expressions of his teachers. He follows the style of the college textbooks. He wrote praiseworthy articles for the student engineering magazine, talks for student groups, and term papers. He never noticed much connection between the principles taught in the English department and those practiced in the science departments.

Students are natural imitators. They may not take seriously what their parents and teachers say, but they imitate what they do. Students' expressions are copied to some extent from the comic strips, the sports pages, and the radio announcers. They must read and hear a considerable amount of correctly written and accurately phrased American prose to bring their modes of expression up to even a moderately literate plane. The teacher of physics should constantly bear these facts in mind and consciously correlate his work with that of the teachers of English composition and technical writing. For example, at the introductory lecture to the course, in which he points out the connection of physics to mathematics and the other physical sciences, the teacher can include English composition as an allied subject. He should certainly use correct diction, grammar, and form in his written and oral material, and he can mark as incorrect in the student's oral and written assignments the same defects that will be noted by the English teacher.

The Teacher Can Keep Abreast of Modern Trends

In order to make this correlation effective, the physics teacher must adopt correct and approved forms. Forms of spelling, punctuation, and abbreviations are gradually but continually changing; if the teacher is not watchful, he will be marking as incorrect something the English department has just emphasized as the correct form. There is no need for him to delve into the intricacies of complex grammar; there may be no point in his remembering which authorities spell the word "disk" and which "disc," though he should be aware of the existence of such differences. But it goes without saying that a cultivated educator in any field should be careful of:

(*i*) *Grammar*. He should recognize the common grammatical faults and know how to revise sentences to overcome them; examples are dangling gerunds, missing antecedents, and nonidiomatic prepositions.

(*ii*) *Diction*. He should know and practice fine distinctions between words, particularly between nouns and verbs defined in physics. Proper definitions are a specific part of the English course. Once having pointed out the distinction between *velocity* and *speed*, or between *mass* and *weight*, the teacher should observe the distinctions.

(*iii*) *Style*. He should watch authoritative technical and nontechnical current American prose for forms and expressions. He should note the growing tendency toward open punctuation and the adoption by the intelligent layman of technical terms.

(*iv*) *Form*. He should learn and use modern spellings, symbols, abbreviations, and forms of tables, charts, and lantern slides. Committees have worked for years on standardization of units and symbols. The American Standards Association distributes them; the professional magazines print them;[1] they are in the handbooks. The instructor in technical writing drills on them. The student should see the same forms in the textbooks and on the blackboard. He should not see units with periods (correct form cm, ft/sec, hp) nor the plural *s* on a unit, as in 10 lbs., a form that has been obsolete for twelve or fifteen years.

The Teacher Can Insist on Correctly Written Textbooks

The physics teacher can use his influence to improve the quality of the prose in textbooks of physics and engineering. The publishing of college scientific texts is a headache for the publisher, who makes little money unless the sale is phenomenal. Many writers of engineering texts are extremely poor in composition. Because they know enough about the subject matter to write a text, they are often disagreeable about accepting

[1] *Am. J. Physics* 16, 164 (1948).

well-meant suggestions on style and form. It should not be the province of the publishing house to see that the prose is clear and the style correct; it should be the responsibility of the author, and possibly of the college in which he teaches.

When the physics teacher sends requested comments to the publisher of a new textbook, he should comment—favorably or unfavorably—on the grammar, diction, style, and form. Later editions may be issued with the more glaring errors corrected. Why should the student think he need write any better than the professional men who write his textbooks?

Errors or serious inconsistencies between the text and current usage in grammar, diction, style, and form should be reported by both teachers and students and should be corrected in the same manner as actual numerical errors in problems. Properly handled, these corrections need not undermine the student's confidence in the subject matter; the practice should merely emphasize the relation of science and English.

The Teacher Can Serve on Committees to Improve Prose Style

If the teacher of physics agrees that physics and engineering are more intelligible when expressed in correct and approved American prose, he must be willing to add still another committee to his already crowded extracurricular work. As first steps in the practical correlation of theory and practice in the writing of American prose, the following suggestions are offered.

(i) Each college should have a permanent style board, which would be responsible for the setting up and maintaining of a style guide for the college staff. The style sheet or manual need not be lengthy. It could well include administrative regulations about typing and mimeographing material, travel accounts, student assistants, and other information for which there is no more logical repository. In addition, it should furnish reliable information and rules for style and form of general college material as well as lists of authorities for specific fields. It could be of inestimable value both to faculty and to office staffs.

(ii) Each college should arrange for the marking of graduate theses not only for technical content but also for style and form. As the student load decreases to prewar size, perhaps even term papers might be marked by the course instructor and a member of the English staff. Dean Elmer Hutchisson has pointed out[2] in this connection that one of the four goals of college training is the art of communicating with human beings, and that a master's thesis in physics or engineering should be suitable for publication in a professional journal.

(iii) Each college should exercise some faculty supervision over the quality of technical prose in the college engineering journal. To quote Dean Hutchisson again, "the idea has grown up that good writing is frowned upon in science." If he is willing to go over student manuscripts before publication and make constructive suggestions, the physics teacher can do much to alter this viewpoint.

Some Things That Need Attention

Lest any reader think that much ado has been made about nothing, let me point out a few of the places where improvement is possible. We must keep uppermost in mind the welfare of the student. He is in college for only four or five years; he has much information to master and even trivial inconsistencies put an unnecessary burden on him.

Contradictions in subject matter. Far outweighing any matters of form and grammar is, of course, any contradiction in statements of fundamental concepts of physics itself. Even such trivial differences as that between a text that says "positive torques are taken counterclockwise" and a laboratory manual that says "positive torques are taken clockwise" disturb a student. He always has a mental conflict when he has to use one of the notations in a problem. As examples of methods of eradicating such inconsistencies, the following suggestions are offered.

(i) Where honest differences in form exist among authorities, discuss them frankly and point out their relative significance. State clearly what form, concept, or definition is going to be accepted in the course.

[2] In a discussion of "The role of physics in graduate instruction in engineering" at the June 1948 meeting of the AAPT and the ASEE in Austin, Texas.

(ii) Explain that even after definitions have been clarified, teachers and students may slip and misstate terms. Permit correction as for a missing square-root sign. Such corrections even during the lecture period prevent mistakes in classroom notes. Insignificant as it might appear, it would be preferable to have the entire physics staff agree on the pronunciation of *data*, *apparatus*, and *research*. It is well to remember that 0.3 is *pound* and not *pounds*.

(iii) Thrash out any differences in presentation, concepts, and form at a staff meeting (just as one might agree on the values of π and g to be used in problems) and present a unified front to the student body. If the style manual suggested earlier were used there would, of course, also be uniformity with the mathematics and chemistry staffs.

Material emphasized in English composition. During the same terms that the student is learning mathematics, physics, and chemistry, he probably is being taught the fundamentals of English composition, effective prose writing, and later some form of technical writing. The material that will be reviewed and emphasized in such courses is (i) punctuation, grammar, spelling, and form; (ii) definitions of words; (iii) descriptions of devices and processes; (iv) analysis of ideas and concepts. There is no apparent reason for any discrepancy when the same type of material is used in a science class.

Sources examined specifically to obtain examples of current practice for this paper include recent textbooks of college physics; lectures, quizzes, and examinations in current college physics courses; laboratory manuals and instruction sheets; scientific literature given out at meetings; college engineering magazines; literature from laboratory supply houses and textbook publishers; speeches at scientific meetings; and science articles from such nontechnical periodicals as the *New Yorker*, the *New York Herald Tribune*, the *New York Times*, *Life*, and *Science Illustrated*. In all the material from specifically physics sources, the same violations of fundamental rules of composition occurred again and again; in most of the material from nontechnical sources there were few or no such errors. It behooves the conscientious teacher of physics to take stock and ask himself questions such as the following:

Punctuation. Do I prefer open or closed punctuation? Do I put commas between the subject and predicate? Do I set off conjunctive adverbs with a semicolon? Do I omit punctuation in centered mathematical expressions?

Grammar. Do I use constructions with dangling modifiers? Do I use pronouns with vague antecedents? Do I use plural nouns and singular verbs?

Spelling. Do I use a single or double *l* in words before a suffix beginning with a vowel? Do I write *wave length*, *wave-length*, or *wavelength*? What Latin plurals have I abandoned? Can I spell *supersede* and *procedure*?

Abbreviations. Do I have a copy of the ASA final standard units, symbols, and abbreviations in scientific writing? Do I use the approved forms and teach them to the students?

Tables and graphs. Do I present tables and charts in the current approved form? Are my lantern slides of the type recommended by society program chairmen?

Prose style. Is my style pedestrian, inaccurate, verbose, unemphatic? Can the students instantly spot the topic sentence in each paragraph of my written material? Is my lecture prose a model of effective communication? Does each of my stated definitions have a term, a genus, and a differentia? Do I insist that the "apparatus and procedure" parts of the students' laboratory reports come up to the standards of those in English composition? Do I ever give a "perfect" or "satisfactory" grade to a report that violates the rules of correct composition?

What Can be Done to Increase Correlation between English and Physics?

Should there be any departments of physics in which the student has not been given the best possible training by demonstrating to him how technical ideas are conveyed in clear, concise American prose, some effort might be devoted to developing and maintaining a closer liaison between the physics and the English departments. Some of the following suggestions might prove of value.

(i) At a joint meeting of teachers of science and English composition, minimum standards might be set up for the students' written work. For example, it might be decided that the following were never acceptable: *due to* as a compound preposition; dangling modifiers; *cycles* for *cycles per second*. The following might be pointed out as not being in the best approved usage: *compare to* for *compare with; while* for *although; so* for *thus;* omission of hyphens in compound adjectives; interchangeable use of *which* and *that* for nonrestrictive and restrictive clauses; and the use of *this* with no clear antecedent.

(ii) Close co-operation may be maintained—at least there should be no contradiction—between composition and physics. It might even be advisable to lighten the load of teachers in turn and permit time to audit in both directions; let physics teachers audit English composition and report writing, and English teachers audit physics classes.

This method would be of the highest value only if frank discussion were offered and suggestions taken. The physics staff would learn easily the most common errors in sentence structure and the English staff would learn both terminology and concepts in physics. If one mild suggestion might be put forth, such auditing should start at the *top* of each staff of the persons doing actual teaching and grading, rather than administrative work. Otherwise, we have the lamentable case of the supervisors being on less firm ground than those being supervised.

(iii) Textbooks in physics should be examined for violations of modern prose style and approved forms. Technical content being satisfactory, the one that best exemplifies approved style should be adopted.

(iv) Some sort of standard style manual or book on report writing should be adopted as a department guide.

(v) All lecture material, laboratory instructions, and typed material intended for bulletin board posting or class distribution should be checked for clarity and form.

(vi) Student's work should be marked not only on content but also on prose style.

(vii) An occasional staff meeting might be devoted to methods of correct presentation, possibly with guests from the English department, editorial department, and secretaries who type manuscripts.

Summary

It is the subject matter of a physics textbook or lecture that must be correct; the style and form are secondary. It should be just as easy, however, to present physics material in correct form as in any other. Whenever possible, engineering material should be presented in currently approved engineering style.

A Boner

Problem (adapted from Lemon and Ference):—The armature of a shunt wound motor, at rest, passes a current i_a when a dry cell of ε volts is connected directly to it. The shunt field coil with the same electromotive force passes a current i_f. What is the back electromotive force E_B of this motor when running on a line voltage V_L under a load that requires a total current of I_T amperes?

A student works this problem as follows, assuming that, when the motor is running on the line, the back emf does not affect the armature current. Writing I_A and I_F for armature and field current when the motor is running on the line, he sets

$$I_A/I_F = R_F/R_A, \tag{1}$$

where R_F and R_A are the resistances of the field and armature, respectively. Then, since

$$R_F/R_A = (\varepsilon/i_f)(i_a/\varepsilon) = i_a/i_f, \tag{2}$$

it follows that

$$I_A/I_F = i_a/i_f. \tag{3}$$

An additional equation is obtained by writing

$$I_T = I_A + I_F. \tag{4}$$

Equations (3) and (4) are then solved for I_A and I_F in terms of the given quantities. The results are

$$I_A = I_T i_a/(i_a+i_f) \quad \text{and} \quad I_F = I_T i_f/(i_a+i_f). \tag{5}$$

Then the student uses the well-known relationship:

Electrical Power = Power Input − Heat Losses, leading to the equation

$$E_B I_A = V_L I_T - (R_A I_A^2 + R_F I_F^2). \tag{6}$$

When Eq. (6) is solved for E_B, after unknowns I_A, I_F, R_A, and R_F are expressed in terms of the given quantities from Eqs. (2) and (5), the student obtains

$$E_B = V_L - I_T \varepsilon/i_a + V_L i_f/i_a.$$

The pay-off comes when the professor solves the problem by an unquestionably correct method and obtains the same result!—LEO SEREN and ELNA HILLIARD, *University of Idaho, Moscow, Idaho.*

A Mechanical Model for Demonstrating Subharmonic Resonance

CARL A. LUDEKE
University of Cincinnati, Cincinnati, Ohio

THE purpose of this article is to present a relatively simple mechanical model which can be used to demonstrate the interesting nonlinear phenomenon known as subharmonic resonance. The author has found this model, together with motion pictures of it in operation, a very useful aid in presenting the subject of subharmonic resonance to a graduate class in nonlinear mechanics. The 16-mm film which was taken at the rate of 128 frames per second with exposures of 1/360th of a second slows down the phenomenon so that the higher order subharmonics can easily be observed. This film, which depicts ordinary resonance and subharmonic resonance of order 1/2, 1/3, 1/5 and 1/7, is available, at a nominal cost,[1] to anyone interested.

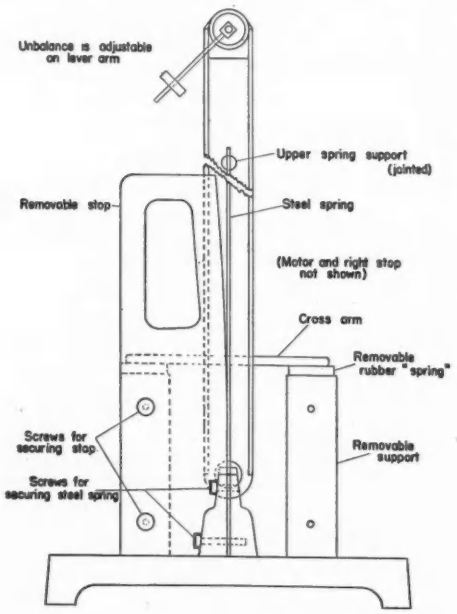


FIG. 1. Front view of model.

[1] The cost of making a copy of approximately 350 ft of 16-mm film.

Theory

It is generally recognized that most physical relations are not quite truly linear, although, for mathematical simplicity, they are usually represented by linear differential equations. This idealization is entirely satisfactory as long as the resulting solutions, which are greatly simplified by this linearization, give results in agreement with experiment. However, there are some phenomena, such as subharmonic resonance and self-excited oscillations, which are characteristic of nonlinear systems; and any attempt to linearize the equations which describe them results in equations whose solutions exclude the very phenomena which we wish to study. Consider, for example, the problem of finding periodic solutions to the equation,

$$m(\mathrm{d}^2x/\mathrm{d}t^2)+f(x)=P_0\sin\omega t. \tag{1}$$

If $f(x)=kx$ so that Eq. (1) is linear, we have the well-known case of linear forced vibrations and the only periodic solution is

$$x=\frac{P_0}{k-m\omega^2}\sin\omega t. \tag{2}$$

If $f(x)=kx+g(x)$, where $g(x)$ is not linear, and $g(x)/kx\ll 1$, we have a quasi-linear equation, one of whose periodic solutions is

$$x=\frac{P_0}{k_e-m\omega^2}\sin\omega t, \tag{3}$$

in which k_e is an equivalent spring constant, which is not actually a constant, but is really a function of the amplitude and is determined by the form of $g(x)$. There are several procedures available in the literature[2] for determining k_e. It will suffice here to indicate the form of k_e for a special form of $g(x)$.

[2] For a general survey of the subject of nonlinear mechanics see N. Minorsky; *Introduction to nonlinear mechanics* (J. W. Edwards, 1947). This reference includes the methods of Poincaire, Van der Pol and Kryloff-Bogoliuboff, which can be used in determining k_e.

Therefore, suppose

$$f(x)=k_1x+g(x)=k_1x+k_3x^3; \qquad (4)$$

so that Eq. (1) takes the form[3]

$$m(dx/dt^2)+k_1x+k_3x^3=P_0\sin\omega t. \qquad (5)$$

If we assume a solution in the same form as Eq. (3), namely

$$x=x_0\sin\omega t, \qquad (6)$$

substitute this in Eq. (5), and use a trigonometric identity for $\sin^3\omega t$, we have as a result

$$k_1-m\omega^2+\tfrac{3}{4}k_3x_0^2 -k_3x_0^2\sin 3\omega t/4\sin\omega t=P_0/x_0. \qquad (7)$$

If Eq. (6) is to be true for all values of t, then Eq. (7) must be true for all values of t. The usual procedure of quasi-linear theory[2, 4] is to neglect the term in Eq. (7) which contains t, thus making Eq. (6) a solution good for all values of t provided x_0 is given by solutions to the cubic equation

$$3k_3x_0^3+4(k_1-m\omega^2)x_0-4P_0=0. \qquad (8)$$

For the differential equation represented in Eq. (5) this procedure is possible provided

$$k_3x_0^3/k_1x_0\ll 1. \qquad (9)$$

However, Eq. (6) is not the only periodic solution of Eq. (5); for it is not necessary that the periodic solution have the same frequency as the applied force. Consider, for example, a solution of the

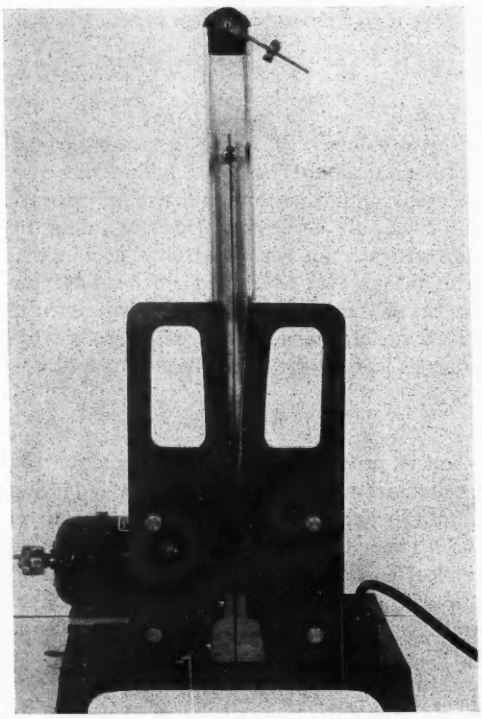


FIG. 2. Front photograph of model.

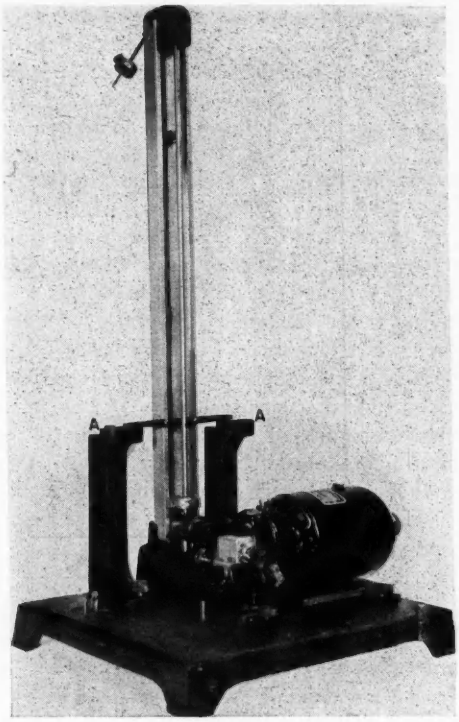


FIG. 3. Photograph showing rubber "spring" system.

[3] This equation was first investigated by G. Duffing and is often referred to as "Duffing's" equation.

[4] For a detailed mathematical discussion see reference 2, or for a slightly different treatment K. O. Friedricks—J. J. Stoker—*Introduction to nonlinear mechanics* (8 lectures)—incorporated in the set of twenty lectures organized under the Program of Advanced Instruction and Research in Mechanics at Brown University and delivered in the academic year 1942–43.

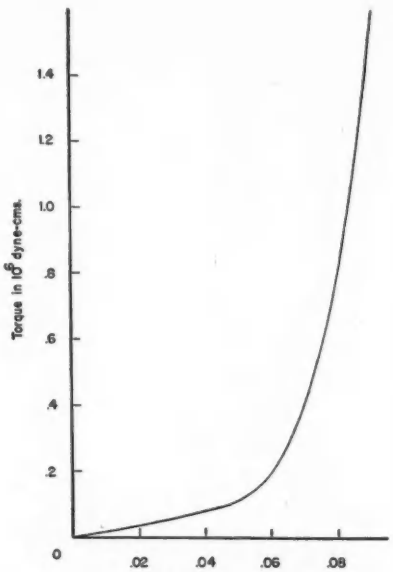


FIG. 4. Load-deflection curve, steel spring with stops.

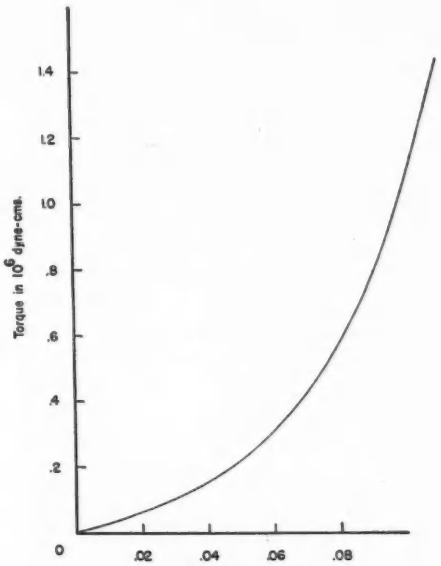


FIG. 5. Load-deflection curve, rubber springs.

form

$$x = x_0 \sin\alpha t, \quad \text{where} \quad \alpha = \omega/3. \tag{10}$$

Substituting this in Eq. (5) we have:

$$(-m\alpha^2 + k_1 + \tfrac{3}{4}k_3x_0^2)x_0 \sin\alpha t - \tfrac{1}{4}k_3x_0^3 \sin 3\alpha t = P_0 \sin 3\alpha t \tag{11}$$

which can be satisfied for all values of t provided

$$\alpha^2 = \frac{k_1}{m} + \frac{3}{4}\frac{k_3}{m}x_0^2, \tag{12}$$

and

$$P_0 = -\tfrac{1}{4}k_3x_0^3. \tag{13}$$

Since this is not intended to be a mathematical treatment[4] of the subject it suffices to say that since Eqs. (12) and (13) can be satisfied simultaneously, Eq. (10) is a possible solution of Eq. (5). It should be noted that linearization of Eq. (5) would eliminate this solution. This incomplete mathematical discussion indicates that if $f(x)$ in Eq. (1) is approximated by a power series of x, and a solution of the form $x = x_0 \sin\omega t/n$ is assumed; then it is possible, under certain conditions, for this solution to exist for some of the values of $n = 1, 2, 3, \cdots$. If, when expanded, $f(x_0 \sin\omega t/n)$ contains all powers of $\sin\omega t/n$, then it is theoretically possible to have solutions for all integral values of n. However, a more complete mathematical analysis,[2,4] including the case of damping, indicates that subharmonic vibrations can exist only if the damping is less than a certain critical value which decreases as n increases. This means that the lower order subharmonics are the ones most easily obtained experimentally. In the model to be described, subharmonics of the order as high as $\frac{1}{10}$th have been observed.

Apparatus

Let us proceed now to the construction of the mechanical analogue. This model was developed from a model presented in a paper several years ago.[5] The vibrating system consists of a small vertical I beam pivoted on its lower end and restored to a vertical position by means of a steel cantilever spring fastened rigidly at its lower end. See Figs. 1 and 2. This spring, which supplies the restoring force, is in itself quite linear. Therefore to introduce nonlinearity into the restoring force, curved stops are placed on each side of the spring. These tend to shorten the effective length

[5] C. A. Ludeke, *J. App. Physics* **13**, 418–423 (1942).

of the spring as it bends to either side, and thus serve to increase the spring "constant." The stops, which are made of wood, and can therefore be easily cut to any shape, are removable so that more than one shape can be utilized in a demonstration. In order to show that nonlinearity can be produced by other means, the stops and spring can be removed and replaced by two rubber pads, as at A in Fig. 3, which become stiffer as they are compressed. This supplies the same type of restoring force as the steel springs and the stops. The actual load-deflection curves are shown in Figs. 4 and 5. The applied force which corresponds to the right side of Eq. (1), is produced by a rotating unbalance mounted at the top of the beam. The driving mechanism is shown in section in Figs. 6 and 7. The most critical feature is the alignment of the center of the universal joint on the center of rotation of the beam. A set of change gears is used so that a clicking device can be utilized to give an audible sound for each cycle of the beam. Thus, for a subharmonic of the order $\frac{1}{3}$, a 3 to 1 gear ratio is used, and therefore a click is heard for every three cycles of the unbalance, which corresponds to every cycle of the beam. A counter is also incorporated so that the frequency can be measured by means of a stop watch. The speed of the slow shaft of the driving motor is between 50 and 100 rev min^{-1} and is held to any of the speeds in this range by a governor which is adjustable while the motor is running.

In a previous model,[5] a small motor was mounted directly on top of the vertical cantilever spring and therefore the gears, universal joint, and long shaft were unnecessary. However, the motor utilized at that time did not have sufficient torque to rotate the unbalance at a uniform angular speed when the beam was in resonance. When in resonance considerable energy is consumed, and the driving mechanism must be such that the motion of the beam does not appreciably effect the uniform angular motion of the unbalance. The present model is satisfactory in this respect.

Under most circumstances the rotating unbal-

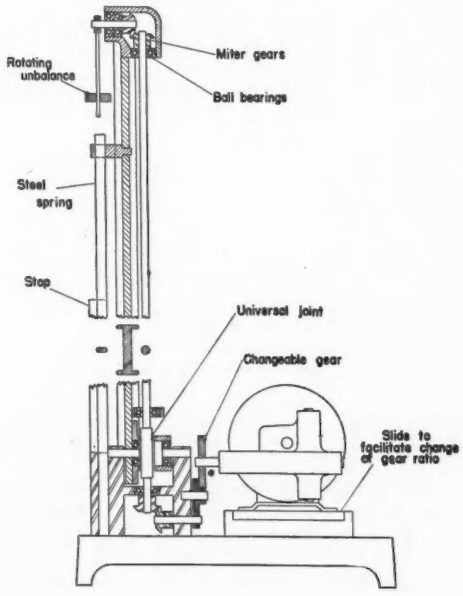


FIG. 6. Side view of model.

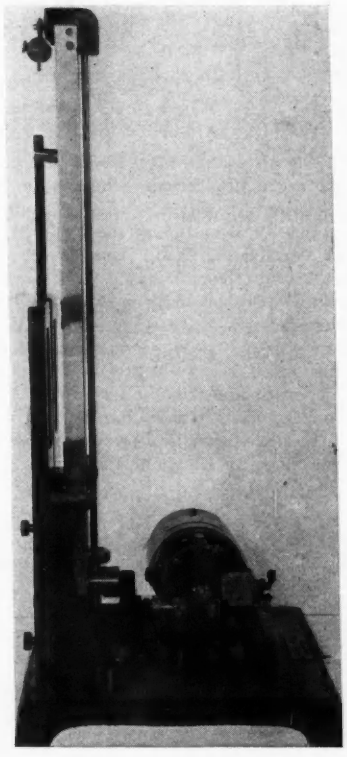


FIG. 7. Side photograph of model.

ance will not start the beam into subharmonic resonance because near the equilibrium position the restoring force is essentially linear. Therefore the model is set in operation by starting the motor and oscillating the beam by hand through fairly large amplitudes until proper conditions are established for sustaining the vibration.

Conclusions

A simple mechanical device has been constructed for demonstrating subharmonic resonance. Its speed of operation is slow enough so that for all but the higher order vibrations, visual counts can be made of the relationship between the frequency of the applied force and the frequency of the maintained vibration. To further aid in such observations a speed film has been made of the model in operation, and is available to those who are interested. Although the theory concerning subharmonic resonance is certainly beyond the range of elementary students there is no longer any reason why the existence of subharmonic resonance cannot be demonstrated to them.

Acknowledgments

The model herein displayed was originally constructed as part of a contract with the Army Air Forces, Wright Field, 1943. The facilities for making the 16-mm speed film were made available by means of a Cottrell Grant-in-Aid from the Research Corporation.

Variable-Resistance Spring Transducer

A highly sensitive mechano-electrical transducer, which transforms slight displacements into large changes of resistance, current, or voltage, is being developed by W. A. Wildhack and his associates of the National Bureau of Standards, Washington, D. C. The active element of the device is a helical or conical spring wound in such a way that the initial tension varies slightly along its length. Thus, when the ends of the spring are pulled apart, the turns separate one by one rather than simultaneously.

When the spring is entirely closed, it has an electrical resistance approximately that of a cylindrical tube. When it is completely open, its resistance is that of the total length of the coiled wire. Resistance can thus be varied over a wide range by stretching the spring. As the percentage change in resistance may be hundreds of times greater than the percentage change in length, displacements as small as 1/100,000 in. can be measured without the use of electrical amplifying devices. The spring transducer thus provides a sensitive means for conversion of any mechanical displacement to a change in an electrical quantity that can be precisely determined. When connected to another transducer which gives a mechanical displacement output (a bimetallic strip responding to temperature changes, for example), the combination gives an easily measurable electrical output. This type of use suggests numerous scientific and industrial applications, including strain gages, pressure elements, accelerometers, electric weighing devices, automatic temperature controls, d.c.–a.c. inverters, and voltage regulators.

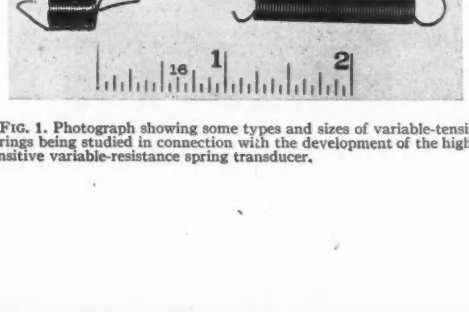


FIG. 1. Photograph showing some types and sizes of variable-tension springs being studied in connection with the development of the highly sensitive variable-resistance spring transducer.

Variation of the initial tension of the spring along its length may be accomplished in several ways: by conical winding, by varying the angle of feed of the wire on a uniform mandrel, or by varying its tension as it is wound. For greatest sensitivity the variation in initial tension is made quite small. To decrease contact resistance between successive turns of the closed spring, a high average initial tension is built into the spring, and the turns are coated with 0.0001 in. of gold. Thus far, nickel-alloy wire has been mainly used, because of its high resistivity and small change of mechanical properties with temperature.

As Fig. 1 shows, it is scarcely possible to tell from their appearance that such springs possess unusual properties.

Three-Dimensional Electrical Units, QLT and ILT

F. W. Warburton
University of Redlands, Redlands, California

NOW that electrical quantities are standardized in absolute units after a long delay as a result of the war, and interest in units is rising again, some of the old difficulties are reappearing and it may be well to present the advantages of modernized three-dimensional cgs and mks systems of units. One difficulty which has long been present is the surprising disregard of the fact that physical meaning and clarity of concept follow closely both the definition and the mode of measurement, so that these two factors of determining a physical quantity need be well coordinated. Certainly this fact emphasizes the need of facing realistically the problem of the "fourth fundamental quantity" *versus* measurement in terms of three primary units.

Ever since the recommendation of the International Commission of Weights and Measures for a return to the absolute system of electrical units was ratified by the General Conference on Weights and Measures in 1933, and the International Committee on Weights and Measures, authorized with this responsibility, met in October 1935 and chose January 1, 1940 as the date for the transfer,[1] teachers and textbook writers have paid increasing attention to the magnetic force of currents. It became known that the standardizing laboratories determine the ampere most accurately by measuring the magnetic force of currents in the current balance. The emphasis on currents[2] received great impetus from the definition of the ampere given in the Consultative Committee of the International Committee of Weights and Measures in September 1935, during the consideration of the mks system advocated by the International Electrotechnical Commission (IEC).

Various authors, in discussing the abrogation of the International System of Electrical Units as distinct from the absolute system, have assumed that the basic absolute system of units is the mks system. They thereby ignore the more specific basic concept of electricity, directly connected with the measurement of charge and current, which is provided by the omission of numerical factors from the Coulomb laws in the electrostatic and electromagnetic systems. The success of these older three-dimensional basic systems is evidence that the mixed concept of an empty space having the properties of dielectric constant and permeability is quite unnecessary. It is well to remember also that the decision to substitute the absolute system for the International System was made by the Committee on Weights and Measures, while the IEC shortly thereafter rejected Giorgi's recommendation for the international ohm, and did not agree on the fourth supposedly fundamental unit. Two important committees protested the request of the IEC, that they choose for the fourth fundamental unit one of the seven practical electrical quantities in use, and voted instead for the "permeability of space," 10^{-7} henry per meter, as the connecting link between the mks system and the internationally accepted electromagnetic system. The Consultative Committee[2] reluctantly named first the ohm and then, on the next day, the ampere as perhaps the most suitable if the choice were to be restricted to one of these seven. The S.U.N. Committee of the International Union of Pure and Applied Physics[3] took the "opportunity of placing on record their recognition of the fact that there are important electrical theories supported by a number of physicists in accordance with which $A=c$ the velocity of wave propagation and $\mu_0 k_0$ is a pure number." This is distinct recognition that electrical theory can be three-dimensional, since $A^2/\mu_0 k_0 = c^2 (= D^2 a/b$ in our notation) is the square of the velocity of light in free space. Yet the courtesy and aid these two leading scientific organizations gave the IEC in adjusting its ideas to the absolute system have been widely misinterpreted as agreement that a "fourth unit" is theoretically necessary. The

[1] *Nature*, **136**, 728 (1935). Kennelly, "IEC adopts mks system of units," Appendix I, II, *Trans. AIEE*, **54**, 1379 (1935).

[2] See reference 1, pp. 1382, 1383.

[3] Kennelly, *Journal of Engineering Education* **27**, 305 (1936).

papers reporting the selection by the Committee on Weights and Measures of January 1, 1948 as the new date for the general adoption of the absolute system[4] specify the cgs system as well as the mks system, and emphasize that they are derived from the primary mechanical units of length, mass, and time.

One factor which apparently acts as a deterrent to acceptance of a three-dimensional electrical system of units to match the measurement of electrical quantities in terms of length, mass, and time, is the desire that electricity be not mechanized and made subservient to mechanics. Another factor is the belief that electrical quantities should be so defined that they will not require electrical quantities to have different combinations of fractional powers of fundamental quantities in different systems of electrical units. A third factor is the reluctance to revise and even to discard some of the older concepts—a procedure which is disagreeable, but necessary, if any real progress is to be made in the process of integrating them thoroughly with new concepts.

The importance of electrical charge as a fundamental[5] electrical quantity has been emphasized by use of tables of units of electrical quantities expressed in terms of length, mass or power, time and charge, MLTQ or PLTQ.[6] This importance can be emphasized still more by making mass subservient in a three-dimensional QLT or ILT system. The three-dimensional system is complete in itself. Furthermore it is capable of expressing mass as a measure of internal energy. Thus, the reason for the fractional powers of MLT becomes evident, and the apparent difficulty really becomes an advantage in leading to a QLT or to an ILT system. Unity of concepts in physics and the intimate connection of mechanics and electricity in the mass-energy relation are enhanced when we let the units work for us, checking these relations in the QLT and ILT systems.

To keep concepts clear by rigorous logic at the risk of becoming pedantic let us carefully follow the simple rule,[7] that of N physical quantities, if p of them are primary quantities with units chosen independently and arbitrarily, then the remainder, n or $N-p$, is the number of equations each defining one new physical quantity in terms of others already known. The procedure outlined below in some detail and the development of Table I may be used as a laboratory exercise on systems of units and definitions of electrical quantities.

The metric standard of length is that of the meter bar kept in a vault in the Park of St. Cloud, in Sèvres, near Paris. The two equations, $\mathbf{A}=\mathbf{a}\times\mathbf{b}$ and $v=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$, define area and volume. Time is not defined in terms of anything else but is chosen as another primary quantity, and its unit, the second, is taken arbitrarily as 1/86,400 of the mean solar day. Two more equations $\mathbf{v}=\mathrm{d}\mathbf{s}/\mathrm{d}t$ and $\mathbf{a}=\mathrm{d}\mathbf{v}/\mathrm{d}t$ define velocity and acceleration. Thus far we have $N=6$ physical quantities, $p=2$ primary quantities, length and time, and $n=4$ defining equations.

To illustrate the fact that the number of primary quantities p is arbitrary,[7,8] consider the possible definition of mass by means of the gravitational attraction of one body m to another body M. The gravitational force may be written without any gravitational constant in the form $F=ma=mM/r^2$, whence $M=ar^2$ so that the mass M becomes measured in meter^3/second^2. Then force is expressed in meter^4/second^4, and all other physical quantities likewise become measured in terms of these two primary units, the meter and the second. This unit of mass, equal to 15×10^9 kg, is not of convenient size and has not been used. However one could choose a unit of mass, one thousand millionth as great as this, so that it would be of convenient size, and call it, say, the kam, but he still would be employing the defining equation, $M=10^9ar^2$, and would retain the two-dimensional system. The mass of the earth would then be expressed as $M=(9.80\ \mathrm{m/sec^2})\ (6.37\times10^6\ \mathrm{m})^2\times10^9=3.97\times10^{23}$ kams $(=3.97\times10^{23}\ \mathrm{mm^3}/$

[4] *Nature* **159**, 325 (1947). *Rev. Sci. Inst.* **18**, 328 (1947).

[5] The use of the word *primary* to designate the units whose values are chosen arbitrarily to build up a system of measurements, as distinct from the use of the word *fundamental* as a base for definitions in a conceptual structure of the universe, was suggested by Professor R. C. Tolman in a letter to the writer several years ago.

[6] Eshback, *Handbook of Engineering Fundamentals*, p. 3–08. Campbell, *Bulletin of the National Research Council* **93**, 69 (1933). Moon, *Am. J. Physics* **16**, 27 (1948).

[7] Abraham, *National Research Council Bulletin* **93**, 16 and 28 (1937).

[8] AAPT Committee Report, *Am. J. Physics* **6**, 150 (1938).

TABLE I. Dimensions of electrical quantities.

Physical quantity	Defining equation	Dimensions MLT mks cgs	Dimensions QLT esu basic*	Dimensions ILT mks, emu basic*	Name in mks	Order of definition with current as 3rd primary quantity basic	Order of definition with current as 3rd primary quantity mks emu
Length	—	L	L	L	meter	1	1
Area	$\mathbf{A}=\mathbf{a}\times\mathbf{b}$	L^2	L^2	L^2	meter^2	2	2
Volume	$v=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$	L^3	L^3	L^3	meter^3	3	3
Time	—	T	T	T	second	4	4
Velocity	$\mathbf{v}=d\mathbf{s}/dt$	LT^{-1}	LT^{-1}	LT^{-1}	meter/second	5	5
Acceleration	$\mathbf{a}=d\mathbf{v}/dt$	LT^{-2}	LT^{-2}	LT^{-2}	meter/second^2	6	6
Mass	—	M	—	—	kilogram		
	F/a	—	$Q^2L^{-2}T^2$	$I^2L^{-1}T^2$	kilogram	10	10
Force	$\mathbf{F}=m\mathbf{a}$	MLT^{-2}	Q^2L^{-2}	I^2	newton		
	$\mathbf{F}=\mathfrak{I}\mathfrak{l}\times\int\mathfrak{I}'d\mathfrak{l}'\times\mathbf{r}/10^7r^3$	—	—	I^2	newton		8
	$\mathbf{F}=i\mathfrak{l}\times\int i'd\mathfrak{l}'\times\mathbf{r}/r^3$	—	Q^2L^{-2}	I^2		8	
	$\mathbf{F}=qq'\mathbf{r}/r^3$	—	Q^2L^{-2}	I^2		(8)	
Work	$W=\int\mathbf{F}\cdot d\mathbf{s}$	ML^2T^{-2}	Q^2L^{-1}	I^2L	joule	11	11
Power	$P=dW/dt$	ML^2T^{-3}	$Q^2L^{-1}T^{-1}$	I^2LT^{-1}	watt	12	12
Charge	$\mathbf{F}=qq'\mathbf{r}/r^3$	$M^{1/2}L^{3/2}T^{-1}$	Q	IL		9	
	$Q=It$	$M^{1/2}L^{1/2}$	—	*IT	coulomb		9
	—	—	Q	IL		(7)	
Current	$\mathbf{F}=\mathfrak{I}\mathfrak{l}\times\int\mathfrak{I}'d\mathfrak{l}'\times\mathbf{r}/10^7r^3$		*QT^{-1}	I	ampere		
	—		QL^{-1}	I	ampere	7	7
	$\mathbf{F}=i\mathfrak{l}\times\int i'd\mathfrak{l}'\times\mathbf{r}/r^3$		QL^{-1}	I		(9)	
Electric field	$\mathbf{E}=\mathbf{F}/Q$		QL^{-2}	*IT^{-1}	volt/meter newton/coulomb	13	13
Electric moment	$\mathbf{p}=Q\mathbf{l}$		QL	*ITL	coulomb-meter	14	14
Polarization	$P=p/v$		QL^{-2}	*ITL^{-2}	coulomb/meter^2	15	15
Source density, "displacement"	$D=Q/A$		QL^{-2}	*ITL^{-2}	coulomb/meter^2	16	16
Space permittivity	$\epsilon_0=(D/E)_0=10^7/4\pi c^2$		—	—	—		17
Potential	$V=W/Q(\therefore V=c^2\int dQ'/10^7r)$		QL^{-1}	*ILT^{-1}	volt	17	18
Capacitance	$C=Q/V$		L	*$L^{-1}T^2$	farad	18	19
Resistance	$R=V/I$		*$L^{-1}T$	*LT^{-1}	ohm	19	20
Inductance	$L=V/(dI/dt)$		T	*L	henry	20	21
Magnetic vector B, flux density	$\mathbf{F}=\mathfrak{I}\mathfrak{l}\times\mathbf{B}$		QL^{-2}	IL^{-1}	riemann newton/amp. meter	21	22
Magnetic moment	$M_B=IA$ $(\mathbf{T}=\mathbf{M}_B\times\mathbf{B})$		QL	IL^2	ampere-meter^2	22	23
Intensity of magnetization	$I_B=M_B/v=n_ai_a/l_a$		QL^{-2}	IL^{-1}	amperian current per meter	23	24
Source intensity	$H=NI/l$		QL^{-2}	IL^{-1}	amp. turn/meter	24	25
Space permeability	$\mu_0=B_0/H_0=4\pi/10^7$		—	—	—		26
Vector potential A	$W=-\mathfrak{I}\mathfrak{l}\cdot\mathbf{A}(\therefore\mathbf{A}=\int\mathfrak{I}'d\mathfrak{l}'/10^7r)$		QL^{-1}	I	joule/amp. meter	25	27
Current source, m.m.f.	$mmf=NI$		QL^{-1}	I	ampere-turn	26	28
Magnetic flux	$\phi=\int\mathbf{B}\cdot d\mathbf{A}$		Q	IL	weber riemann-meter^2	27	29

* Dimensions marked with an asterisk (*) are for esu or for mks, emu only. The parentheses (7), (8), and (9) indicate the order of definition when charge is the 3rd primary quantity.

sec^2), in place of 5.97×10^{24} kg or 3.97×10^{14} m^3/sec^2.

Turning now from the astronomical scale of distances to the atomic scale, and to the concept of mass as a measure of the internal electrical energy in a molecule or in a particle, we may enquire why we cannot define mass in terms of charge after having specified the electrical charge in terms of an arbitrary standard. The answer is that we cannot lay away a charged electroscope, for example, in a vault, and have any assurance that the charge and the deflection of the electroscope will remain constant and determined with sufficient precision when we compare other charges with it from time to time. We must first define mass and force and then define charge or current, even if mass be considered a primary quantity only temporarily, or else we must use an equivalent indirect method. The metric standard of mass is the prototype kilogram bar preserved in Sèvres. The newton of force is defined as the product of mass in kilograms and acceleration in meter/second^2, $\mathbf{F}=m\mathbf{a}$, which gives the meaning of force as that which produces acceleration of a body. Note that the physical concept of force would become somewhat more concrete if the agency (body) exerting the force were specified as well. The joule of work is the scalar product of force and distance, $W=\mathbf{F}\cdot\mathbf{s}$, and power in watts is the ratio of the work done to the time required for its performance, $P=\mathrm{d}W/\mathrm{d}t$. The physical meaning of each of these fundamental quantities, force, work, and power, follows directly from its definition. Adding the definition of electric charge in terms of force and distance, Eq. (1) below, we have, thus far, $N=11$ quantities, $p=3$ primary quantities, length, mass and time, and $n=8$ defining equations. In each case the defining equation represents directly the measurement made in determining the quantity. If, in place of charge, we define current in abamperes as the eleventh physical quantity, the eighth defining equation, Eq. (2) below, represents directly the measurement made in the current balance, determining the size of the abampere in absolute units.

Electricity is defined in terms of the forces it exerts. In the electrostatic system the unit of electrical charge is the statcoulomb. It is that charge which repels a like charge with a force of one dyne at a distance of one centimeter,[9] according to the defining equation,

$$\mathbf{F}=qq'\mathbf{r}/r^3. \tag{1}$$

Note that Eq. (1) is the functional definition of electrical charge and gives the physical meaning of electricity as that entity which exerts an electrostatic force on another entity like it. The charge q then has units $\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{3}{2}}\mathrm{T}^{-1}$ in terms of length, mass and time. The fractional exponents indicate that something is amiss, yet it is not correct to insist, as some authors do, that a fourth fundamental quantity, the permittivity of free space ϵ_0, is necessary. Rather than invite an impasse by "charging off" the difficulty to the unknown "properties of free space," let us face the question and see what the dimensions are trying so hard to tell us—that charge is more fundamental than mass. According to Eq. (1), force has units $F=\mathrm{Q}^2\mathrm{L}^{-2}$ in terms of q, while energy has units $W=\mathrm{Q}^2\mathrm{L}^{-1}$, and mass has dimensions $m=\mathrm{Q}^2\mathrm{L}^{-3}\mathrm{T}^2=(\mathrm{Q}^2\mathrm{L}^{-1})/(\mathrm{L}^2\mathrm{T}^{-2})$ which are the dimensions of W/c^2, the mass equivalent of energy. That is to say, the mass m which is the outward characteristic of a body as a whole not only is proportional to W, but has the dimensions of W/c^2, where W is the mutual internal energy of the electrical charges q making up the body.

If we dislike changing the primary unit from mass to charge in the middle of the system, we can go all out for the QLT system by choosing the size of the primary unit of charge in a somewhat indirect fashion before mass and force are defined. Consistently with Eq. (1), the statcoulomb is that charge which when one centimeter away from a like charge attached to a body, like the one kept in Sèvres but 1/1000 as great, will give the second charge and the body attached to it an acceleration of 1 cm/sec^2. This method, which at first seems so bizarre, is not without its advantages, for with electric charge one of the three

[9] Although it is common usage to specify the position of the two charges *in vacuum*, a little consideration shows this restriction should be omitted for it implies erroneously that the force of q' on q changes when a dielectric is brought near. The theory of polarized dielectrics is based on the force of q' on q remaining the same, $qq'\mathbf{r}/r^3$, while the polarization charges q_p in the dielectric medium, likewise obeying $qq_p\mathbf{r}/r^3$, also act on q giving a net reduced force on q. When all the polarization charges q_p are absent, so that one measures only the force of q' on q, he is of course performing the experiment in vacuum, but Eq. (1) is valid whether or not q_p is removed.

primary quantities, force is then defined by the Coulomb law, Eq. (1), and has physical meaning as that which one charge q' does to another charge q while accelerating it. (This is indicated in Table I by definition (8). See note at end of table.) Then mass, that conceptual bugbear of elementary teaching, is the outward aspect of the internal energy of a body defined by $m=F/a$. We are assuming here that all force may be treated as electrical in nature in the sense that the mutual magnetic force of two magnets, of a magnet and a current, or of two currents, is that part of the electrical force which is due to the relative velocities of the charges and which is not balanced out electrostatically; and likewise, the very small gravitational force of the wires is what is left of the mutual forces of the electrical charges making up the wires, when the magnetic force also is balanced. The necessity for the indirect procedure, when charge is defined first, is not that electrical charge is any the less fundamental, but simply that it is easier to lock up a block of inert matter in a vault than it is to split off an active primary ingredient of the matter, a number of electrons measured in statcoulombs, and lock them up alone in the vault.

Now the flow of electrons through the wires of the current balance can be controlled more accurately than can the number of electrons on an electroscope or absolute attracted disk electrometer. Thus we come to the functional definition of the abampere, as that current which exerts and experiences the force given by

$$\mathbf{F}=i\mathbf{l}\times\int i'\mathrm{d}\mathbf{l}'\times\mathbf{r}/r^3. \qquad (2)$$

This current, and the ampere likewise, has dimensions $i=F^{\frac{1}{2}}=\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}\mathrm{T}^{-1}$. Conversely the units for force are given by $F=\mathrm{I}^2$, hence those of energy become $W=\mathrm{I}^2\mathrm{L}$ while mass has the dimensions $\mathrm{I}^2\mathrm{L}^{-1}\mathrm{T}^2$, which are the dimensions of W/c^2, $(\mathrm{I}^2\mathrm{L})/(\mathrm{L}^2\mathrm{T}^{-2})$. These dimensions differ by a factor of $(\mathrm{LT}^{-1})^2$ from those of mass expressed in statamperes because the magnetic force itself in Eq. (2) is a function of the velocity c of propagation of electric effect from current i' to current i.

In order to see how the numerical factor in the mks system of units was chosen dependent on the cgs system, to appreciate the role that the speed of light plays in any system of units, and to find the relationships between quantities in different systems, one may combine Eq. (1) and Eq. (2). One writes first the general form for the total electrostatic and magnetic force exerted simultaneously on a moving charge, as is the case, for example, when measurements are made in crossed fields on the ratio e/m of an electron, proton, or other charged particle. The combined electrostatic and magnetic force has the magnitude

$$F=aq\int \mathrm{d}q'/r^2+bil\sin\alpha\int i'\mathrm{d}l'\sin\theta/r^2$$
$$=qE+Il\sin\alpha B, \qquad (3)$$

where

$$\mathbf{E}=a\int \mathrm{d}q'\mathbf{r}/r^3 \quad \text{and} \quad \mathbf{B}=b\int i'\mathrm{d}\mathbf{l}'\times\mathbf{r}/r^3,$$

and where a and b are numerical constants which depend on the system of units chosen. The constants are not independent, however, and the value of a depends on the value chosen for b and *vice versa*. Using the relation expressing a moving charge $\mathrm{d}q$ as a current in a short length $\mathrm{d}l$ of a wire,

$$Di\mathrm{d}l=v\mathrm{d}q, \qquad (4)$$

where D is a constant which has no meaning when $i=\mathrm{d}q/\mathrm{d}t$ and in this case can formally be set equal to unity and dimensionless, Eq. (3) may be written as

$$F=aq\int\frac{\mathrm{d}q'}{r^2}\left[1+\frac{v\sin\alpha}{D}\frac{b}{a}\frac{v'\sin\theta}{D}\right]$$
$$=aq\int\frac{\mathrm{d}q'}{r^2}\left[1+\frac{v\sin\alpha\, v'\sin\theta}{c^2}\right], \qquad (5)$$

where the ratio D^2a/b is replaced by another constant c^2. It is clear now that c must be a velocity in order that the two terms which are added together may have the same units. Measurement of c in the experiment determining the capacitance of the parallel plate condenser (where the capacitance is given by $C=(10^7/c^2)(A/4\pi d)=\epsilon_0 A/d$ farads), shows that it is the velocity of light, and confirms the conclusion, mentioned above, that magnetic force is the modification of electrostatic force resulting from the velocities of the charges

relative to the velocity of the electric effect through space from one charge to the other.

The Maxwell field equations—obtained by taking the surface and line integrals about the fields of the charge and current of Eqs. (3) and (4), and reducing the volumes and surfaces enclosed to elements in which the charge density is ρ and the current density is $\mathbf{j}$—take the forms,

$$\text{div}\mathbf{E}=4\pi a\rho/\kappa,\ \text{curl}\mathbf{E}=-(1/D)(\mathrm{d}\mathbf{B}/\mathrm{d}t), \quad (6)$$
$$\text{div}\mathbf{B}=0,\ \ \text{curl}(\mathbf{B}/\mu)=4\pi b\mathbf{j}+(\kappa b/Da)(\mathrm{d}\mathbf{E}/\mathrm{d}t),$$

where κ and μ are the dimensionless dielectric constant and permeability characteristic of the medium (each equal to unity in vacuum) and independent of the transformation constants a, b and D. Using the results of Eqs. (5) and (6) the wave equation becomes

$$\nabla^2\mathbf{E}=\frac{\mu\kappa b}{D^2a}\frac{\mathrm{d}^2\mathbf{E}}{\mathrm{d}t^2}=\frac{\mu\kappa}{c^2}\frac{\mathrm{d}^2\mathbf{E}}{\mathrm{d}t^2} \quad (7)$$

and has the specific form in terms of the velocity c which is the same for all systems of units.[10] One needs to exercise care not to attribute too much meaning to the constants a, b and D, for general dimensional constants in electromagnetic theory permit diverse concepts to be brought together without clear differentiation between them. The relation $D^2a/b=c^2$, obtained by simple dimensional analysis, already eliminates one of these constants while the physical definitions chosen determine the values of the two remaining constants and hence the system used.

In the electrostatic system, Eq. (1) is the defining equation, and no factor a should be included since each defining equation specifies one and only one new physical quantity in terms of the others. This condition may be expressed formally by setting $a=1$ and considering it to be dimensionless, so that $b=1/c^2$, provided also that $D=1$ and is dimensionless to insure that current be expressed in statamperes, $i=\mathrm{d}q/\mathrm{d}t$. The circuital equations become $\text{curl}\mathbf{E}=-\mathrm{d}\mathbf{B}/\mathrm{d}t$ and $\text{curl}(\mathbf{B}/\mu)=4\pi\mathbf{j}/c^2+(\kappa/c^2)\mathrm{d}\mathbf{E}/\mathrm{d}t$.

In the electromagnetic system the chosen defining equation, Eq. (2), does not include b. This is expressed formally by setting $b=1$ and dimensionless. Then $a=c^2$, if also D is unity and dimensionless to insure that $q=\int i\mathrm{d}t$, and the electric and magnetic force, Eq. (3), becomes

$$F=q\int c^2\mathrm{d}q'/r^2+il\sin\alpha\int i'\mathrm{d}l'\sin\theta/r^2, \quad (8)$$

with i in abamperes and q in abcoulombs. Also $\text{curl}\mathbf{E}=-\mathrm{d}\mathbf{B}/\mathrm{d}t$ and

$$\text{curl}(\mathbf{B}/\mu)=4\pi\mathbf{j}+(\kappa/c^2)\mathrm{d}\mathbf{E}/\mathrm{d}t.$$

The *basic* cgs system adheres more closely to usage, for it is like the electrostatic system in electrostatics and like the electromagnetic system in magnetics. It is expressed explicitly by setting both $a=1$ and dimensionless and $b=1$ and dimensionless, utilizing the functional definitions both of charge and of current. Then by Eq. (5), $D=c$, $i=(1/c)\mathrm{d}q/\mathrm{d}t$ and

$$F=q\int\mathrm{d}q'/r^2+il\sin\alpha\int i'\mathrm{d}l\sin\theta/r^2 \quad (9)$$

with q in statcoulombs and i in abamperes, while Eqs. (6) yield $\text{curl}\mathbf{E}=-(1/c)\mathrm{d}\mathbf{B}/\mathrm{d}t$ and $\text{curl}(\mathbf{B}/\mu)=4\pi\mathbf{j}+(\kappa/c)\mathrm{d}\mathbf{E}/\mathrm{d}t$.

The constant in the wave equation given by Glazebrook,[11] by Birge[12] and in this paper is expressed, in this same order, as $c^2=A^2/\mu_0k_0=k^2ab=D^2a/b$ with $A=k=D$, $k_0=1/a=1/a$ and $\mu_0=1/b=b$ respectively. A and k, however, were identified in the Gaussian system with the wave velocity c but were not identified with the same factor c, relating the abampere to the statcoulomb, because setting $\mu_0=1/b=1$ in the Coulomb law for fictitious magnetic poles leaves the unit of current undefined while specifying that the unit of magnetic field in the Gaussian system is identical with that in the electromagnetic system. The Gaussian system retains the statampere and the statohm, which means that $D=a=1$, $b=1/c^2$ in Eq. (3) and in Eq. (5) like the electrostatic system, while it requires an inconsistent partial shift to $a=b=1$, $D=c$ by the time it arrives at Eqs. (6). When one focusses attention on the physical concepts and is specific throughout, using currents in Eq. (2), rather than magnetic

[10] This equation is often written as $\nabla^2\mathbf{E}=\mu\kappa\mu_0\epsilon_0\mathrm{d}^2\mathbf{E}/\mathrm{d}t^2$ with $D=1$ and dimensionless, $\epsilon_0=1/a$ and $\mu_0=b$ in emu or $\epsilon_0=1/4\pi a$ and $\mu_0=4\pi b$ in "rationalized" mks units. See Eq. (11).

[11] Glazebrook, *Proc. Roy. Soc.* **48**, 444 (1936).

[12] Birge, *Am. J. Physics* **3**, 177 (1935).

poles, he arrives at the *basic* system rather than the Gaussian system. This *basic* system has all the advantages of symmetry of the Gaussian system, but much more important than symmetry, the physical meaning is associated with the functional definitions. The electric field E, in dynes per statcoulomb, is the intermediary between q' and q, and the magnetic field vector B, in gauss, is the intermediary between the current i' abamperes which exerts magnetic force on the current i. The use of Eq. (4) with $D=c$ is specific in that it calls attention to the transmission velocity c from i' to i, which profoundly affects the nature of magnetic forces and which is inescapable in dealing with units. Jeans[13] and Mason and Weaver[14] use this relation of current to charge, $i=(1/c)\mathrm{d}q/\mathrm{d}t$.

The mks system is the complete practical system. The ampere was defined long ago by the relation, I amperes $=10i$ abamperes, with Q coulombs $=10q$ abcoulombs, so the volt given by $V=W/Q$(joule/coulomb) is the same as $V=10^{-7}W/Q$(erg/coulomb) and should be nearly the emf of a Daniell cell. Substituting[15] these relations and F (dynes) $=10^5F$ (newtons), in Eq. (8) one obtains

$$F\ (\text{newtons})=Q\int c^2\mathrm{d}Q'/10^7r^2 + Il\sin\alpha\int I'\mathrm{d}l'\sin\theta/10^7r^2, \quad (10)$$

with I in amperes, Q in coulombs, l and r in meters and c in meter/sec, so that in mks units, $a=c^2/10^7$ with the dimensions meter^2/second^2 and $b=10^{-7}$ is dimensionless. The requirement of the three-dimensional mks system that b have the value $b=10^{-7}$ specifies the magnitude of electrical quantities quite as completely as does the choice of $\mu_0=10^{-7}$ or $4\pi/10^7$ as a fourth fundamental unit. Thus the basis for the agreement[16] at the Scheveningen meeting of the E.M.M.U. Committee of the IEC in 1935, regarding the supposed need of a fourth "fundamental" unit, is very weak indeed. The reason given was that "a fourth unit was needed because it would be possible, starting with the three units, meter, kilogram and second, to construct a number of possible associated electromagnetic series, differing from the existing practical series which all desired to maintain."

Many authors prefer to add a factor 4π in the denominator of Eq. (10) and combine the remaining factors in the symbols $\epsilon_0=10^7/4\pi c^2$, the so called "permittivity of free space," and $\mu_0=4\pi/10^7$, the so called "permeability of free space." Then

$$F\ (\text{newtons})=Q\int \mathrm{d}Q'/4\pi\epsilon_0r^2 + Il\sin\alpha\int\mu_0I'\mathrm{d}l'\sin\theta/4\pi r^2. \quad (11)$$

In this form the system is called the rationalized mks system.

If we wish to go all out for the ILT system, we can determine the size of the ampere in an indirect manner, consistent with Eq. (10) applied to parallel wires,

$$F=2I^2l/10^7r. \quad (12)$$

The ampere is that flow of charge in a long straight wire one meter away from an equal current constrained to flow along one edge of a standard unit body (equal to that labeled the kilogram and kept in Sèvres) which will give a one-meter section of the second current and the standard body attached to this section an acceleration of 2×10^{-7} meter/second^2. The newton is then defined as and has the meaning of that

[13] Jeans, *Mathematical theory of electricity and magnetism*, (Cambridge, 1923), p. 511.

[14] Mason and Weaver, *The electromagnetic field* (Chicago, 1932), p. 233.

[15] If the equation involved has precisely the same form in all systems of units and the units are omitted from the equation, the algebraic symbol represents the combination of the numerical value and its unit, and transformation from one system to another takes the form, $Q=1$ abcoulomb $=10$ coulomb, etc. On the other hand when the units are specifically included in the equation, or when the equation has a different form in different systems and the units are stated for each system, the symbol represents only the numerical value of the physical quantity. In this case transformation equations to another system take the form, q abcoulombs $=10^{-1}Q$ coulombs, F newtons $=10^{-5}F$ dynes, etc., where the conversion factors, which are unity, refer to the numerical part. These are algebraic equations which may be solved simultaneously with the specific equation in one system to obtain the corresponding specific equation in the other system.

[16] Kennelly, *Journal of Engineering Education* **27**, 295 (1936).

which one current I' does to another flow of charge I according to Eq. (12), in addition to whatever electrostatic force may be present; and the kilogram mass is defined by $m=F/a$. These functional definitions of current and charge represent the common understanding of electricity as that agency which acting in the motor turns our machinery, which provides heat energy in the resistor by agitation of atoms by means of electrostatic forces, and which combines its electric and magnetic forces in oscillating circuits. In addition their intimate relation with force and mass introduces the modern concept of the equivalence of mass and energy.

Electrical potential is defined as the ratio of the energy W made available and equal to the work done in moving a charge Q, to the charge itself. The defining equation is

$$V=W/Q. \tag{13}$$

The volt can be determined by measurement of the heat energy W liberated in an electric calorimeter by the passage of a charge Q through the heating element, but more precise measurement is made with an inductance. Capacitance is the ratio of the charge on a condenser to the potential, $C=Q/V$, and V in statvolts can be determined in the same experiment with the absolute electrometer, in which charge is measured in statcoulombs and the capacitance computed in centimeters. Resistance is defined by the ratio $R=V/I$ which has long been known as Ohm's Law and in which R is nearly constant for many materials. The coefficient of inductance is defined as the ratio of induced potential to the rate of increase of current, $L=V/(\mathrm{d}I/\mathrm{d}t)$. Thus far we have $N=15$, $p=3$, and each of the $n=12$ defining equations gives the meaning of one and only one new quantity in terms of others already known. The physical concept involved in the definition of potential in Eq. (13) is essential in the definitions of capacitance, resistance and inductance, and the measurement of the volt follows the definition quite closely, but again somewhat indirectly. The value of an inductance is computed from its dimensions, resistance is determined most accurately by comparison with inductance and time, and the value of the volt is found from the measured current in this resistance. The necessity for this indirect procedure is not that electrical potential is any less fundamental than inductance or resistance, but simply that greater precision is attained in the method of balancing the IR potential drop in a resistor against the voltage induced in an inductance.

Table I illustrates the QLT and ILT systems by giving defining equations of common mechanical and electrical quantities and their dimensions in terms of MLT, QLT and ILT. Magnetic moment M_B, (such that the torque is $\mathbf{T}=\mathbf{M}_B\times\mathbf{B}$) which is descriptive of a coil or magnet itself[7, 17] not including permeability of the surrounding medium, is used in preference to M_H, defined by $\mathbf{T}=\mathbf{M}_H\times\mathbf{H}$. In the *basic* system all quantities have dimensions given in both QLT and ILT. Dimensions marked with the asterisk (*) in the QLT column are for esu only, and those so indicated in the ILT column are for mks and emu only. In each case the dimensions in the *basic* system may be obtained from the correct values in the adjacent column by the relation $\mathrm{Q}=\mathrm{IL}$ (since $i=\mathrm{d}q/c\mathrm{d}t$). Resistance in the *basic* system, which does not appear correctly in either column, is dimensionless. In the *basic* system it seems more logical to choose charge as a primary quantity and define force in terms of charge in the Coulomb law, even though the magnetic measurement is the more precise. The order of definition, with charge as the third primary quantity, is indicated in parentheses in Table I wherever it differs from that with current as the third primary quantity.

The field vector $\mathbf{B}$ (weber/m^2, or newton/amp-m) is an important physical quantity and needs a name to correspond to the gauss in the electromagnetic system. Riemann did important work in electricity, along with Gauss, Weber and Ampere. Riemann's formula for the magnetic force of moving charges is derived from energy expressed in terms of the standard vector potential $\mathbf{A}$. The name "riemann" is suggested for $\mathbf{B}$ in mks units.

In this discussion of three-dimensional systems of electrical units we have treated all forces as electric and have assumed that the comparatively small gravitational forces of two bodies will be found to fall into this description. This concept is in accord with the efforts of many physicists to

[17] Stratton, *Electromagnetic theory* (McGraw-Hill, 1941), pp. 235, 239.

provide a unified theory of electricity and gravitation. Exchange forces are electric for they give at present the best statistical description of the relative position of charges affecting the orientation of magnetic moments in the atom. In nuclear physics the coulomb electrostatic law and the classical magnetic force of one moving charge on another are inadequate, and the differences from classical forces may be considered to be variations in the electrical forces themselves, required at these small distances.

When it is necessary to write a general equation containing factors to which different values can be assigned in particular systems of units, Eqs. (3) and (5) with constants a and b are preferred over those general equations involving the constants ϵ_0 and μ_0 as "properties of free space," because the former avoid preconceptions of the old fashioned ether, which have given us so little (and misleading) information for so many years, and because they show that the velocity of electromagnetic propagation c is inevitably present in any complete system. Also Eqs. (3), (5), (8) and (10) show that the one constant factor remaining in the mks force equation is the 10^7 ergs/joule conversion factor. Surely it is no more difficult to carry 10^7 in equations of some systems and not in others, than to carry 4π in Coulomb's law in some systems and not in others, when we know that the 10^7 ergs/joule appears because of the shift to the practical system with the ampere defined in terms of the cgs system, and when the explicit 4π is cancelled by another 4π contained implicitly within ϵ_0 and μ_0. And it is more definite and more satisfying to the analytical mind to use μ_0 as a convenient substitute for $4\pi/10^7$, and ϵ_0 as a convenient substitute for $10^7/4\pi c^2$, than it is to have these constants represent properties of an outmoded ether carried over into the nothingness of free space.

If we are to maintain the leadership in science that is essential to national strength, we must vigorously press ahead in research. There is one simple axiom on which this thought is based. The secrets of nature are not our monopoly. Any nation that is willing and able to make the effort can learn the secrets that we have learned. Such a nation may, indeed, discover new facts of nature we have not yet discovered.

Our problem, therefore, is not a static one of preserving what we have. Our problem is to continue to engage in pure—or fundamental—research in all scientific fields. Such research alone leads to striking developments that mean leadership. Yet it is precisely in this area that we, as a Nation, have been weakest. We have been strong in applied science and in technology, but in the past we have relied largely on Europe for basic knowledge.

Pure research is arduous, demanding, and difficult. It requires unusual intellectual powers. It requires extensive and specialized training. It requires intense concentration, possible only when all the faculties of the scientist are brought to bear on a problem, with no disturbances or distractions.

Some of the fundamental research necessary to our national interest is being undertaken by the Federal Government. The Government has, I believe, two obligations in connection with this research if we are to obtain the results we hope for. First, it must provide truly adequate funds and facilities. Second, it must provide the working atmosphere in which research progress is possible.—PRESIDENT HARRY S. TRUMAN, *at the Centennial Celebration of the* AAAS, September 13, 1948.

A Note on the Significance of Power Reflection

BERNARD SALZBERG
Naval Research Laboratory, Washington 20, D. C.

THE term *intensity of reflection* has been used for many years in writings on optics. In recent years the term *coefficient of power reflection* has come into common usage, particularly in writings dealing with propagation of electromagnetic waves through a stratified medium and along transmission systems such as wave guides and paired conductors. Both of these terms are defined as the ratio of reflected to incident intensity or power. It does not appear to be generally recognized that when propagation takes place in an absorbing medium, or in an attenuating system generally, this definition may no longer be unique and consequently the concept of power reflection has no significance.[1] When propagation takes place in a medium for which the phase angle of the characteristic admittance is zero, corresponding either to a lossless medium,[2] or to an absorbing—but distortionless—medium, the definition is unique. It is also unique when the phase angle of the characteristic admittance is equal to the phase angle of the terminating admittance. Only for these special situations, however, is the definition unique and equal to the square of the absolute ratio of reflected to incident electric or magnetic amplitudes. For all other situations the definition is not unique, and indeed the concept and terminology of incident and reflected power has no real meaning or significance. In contrast, the concept and terminology of *amplitude reflection* may always be used.

To illustrate these remarks, and to indicate in detail the fundamental nature of the limitations which circumscribe the concept of incident and reflected power, the following simple discussion is provided: Consider two adjoining semi-infinite mediums, the surface of separation being plane. The electromagnetic properties of each of these mediums are unrestricted and in general different. A steady-state harmonic plane wave of electric and magnetic amplitudes is supposed to have been set up by a source located in one of these mediums. In that medium there is, in general, an incident wave and a reflected wave. In the other medium there is only a transmitted wave; a reflected wave does not occur in this medium because of its infinite extent. Each of the waves is supposed to vary with time as $\exp(ipt)$, where p represents the angular frequency, $i=(-1)^{\frac{1}{2}}$, and t is the time. Propagation is supposed to take place along the x-direction, normal to the plane interface, located at $x=0$, which defines the boundary between the two mediums.

It will be observed that the foregoing represents a verbal description of the solution of Maxwell's equations appropriate to the assumed situation. Symbolically this solution may be written as follows:

$$E_1=E_i\exp(-P_1x)+E_r\exp(P_1x) \quad (1)$$

$$H_1=H_i\exp(-P_1x)-H_r\exp(P_1x) \quad (2)$$

$$E_2=E_t\exp(-P_2x) \quad (3)$$

$$H_2=H_t\exp(-P_2x) \quad (4)$$

$$E_i+E_r=E_t \quad (5)$$

$$H_i-H_r=H_t. \quad (6)$$

Equations (1) and (2) express the total complex electric and magnetic amplitudes, respectively, in the first medium in terms of incident and reflected components. Equations (3) and (4) represent the complex transmitted electric and magnetic amplitudes, respectively, in the second medium. Equations (5) and (6) provide the relations between values of the incident, reflected and transmitted amplitudes at the interface between the two mediums. The symbols P_1 and P_2 in Eqs. (1)–(6) represent the propagation constants in the two mediums; they are, in general, complex, the real parts being the attenuation constants, and the imaginary parts the phase constants. Corresponding electric and magnetic amplitudes are related by the appropriate

[1] See, for example, J. A. Stratton, *Electromagnetic theory* (McGraw-Hill, 1941), p. 512.

[2] S. A. Schelkunoff, *Electromagnetic waves* (D. Van Nostrand, 1943). See p. 212, where propagation along a lossless transmission line is treated, correctly, in terms of incident and reflected power.

characteristic admittances; like the propagation constants, these admittances are, in general, complex functions of the angular frequency and the electromagnetic properties of the mediums.

$$H_i = A_1 E_i, \tag{7}$$

$$H_r = A_1 E_r, \tag{8}$$

$$H_t = A_2 E_t. \tag{9}$$

A definition for the coefficients of electric amplitude reflection and magnetic amplitude reflection may be obtained from relations (5)–(9). It is

$$r = E_r/E_i = H_r/H_i = (A_1 - A_2)/(A_1 + A_2). \tag{10}$$

Since E and H are in space quadrature, the power transmitted per unit area across the interface into the second medium is given by Poynting's theorem as

$$P_t = \tfrac{1}{2}\mathrm{Re}\,(E_t H_t^*), \tag{11}$$

where Re denotes *real part of*, and * indicates that the conjugate of the complex quantity with which it is associated is to be used. All quantities are expressed in units of the rationalized mks system. By means of Eqs. (5) and (6), (11) may be written as

$$P_t = \tfrac{1}{2}\mathrm{Re}(E_i H_i^*) - \tfrac{1}{2}\mathrm{Re}(E_r H_r^*) - \tfrac{1}{2}\mathrm{Re}(E_i H_r^* - E_r H_i^*). \tag{12}$$

According to usual definitions, the first term on the right-hand side of Eq. (12) is identified as the incident power per unit area, P_i, and the second term is identified as the reflected power per unit area, P_r. The third term consists of cross-products of incident and reflected amplitudes, and cannot therefore be uniquely identified with P_i or P_r. It is clear that unless the latter term is identically zero, conservation of energy, expressed in the form $P_i = P_t + P_r$, where these symbols are defined in the usual way, is not achieved. This difficulty conceivably could be circumvented by generalizing the definitions of P_i and P_r so as to include the components of the third term. Unfortunately, since these components are cross products, there is no unique way to carry out such a generalization. Consequently, except for special situations for which the third term vanishes, the concept and terminology of incident and reflected power has, strictly speaking, no real significance. Fortunately, in many important technological situations the third term is small in comparison with the other terms, so that the concept of incident and reflected power may be used with only small error.

The circumstances in which the third term of Eq. (12) vanishes are readily determined. By means of relations (7), (8) and (10), the third term of Eq. (12) may be expressed as

$$P = |E_i|^2 \,\mathrm{Im}(A_1)\,\mathrm{Im}(r), \tag{13}$$

in which Im denotes *imaginary part of*. Thus, aside from the uninteresting cases for which $A_1 = 0$ or $r = 0$, Eq. (13) can vanish only when A_1 is entirely real or when r is entirely real. The first condition, $\mathrm{Im}(A_1) = 0$, corresponds to the following two special situations: (a) the first medium, in which the source is located, is lossless; or (b) the first medium is an absorbing one, but of a type for which the phase angles of the distributed shunt admittance and distributed series impedance are equal: this is the condition for what Heaviside termed a distortionless medium. The special situation which corresponds to the second condition, $\mathrm{Im}(r) = 0$, can be deduced by inspecting Eq. (10). It occurs when the phase angle of the characteristic admittance A_1 is equal to the phase angle of the terminating admittance A_2.

A definition for the coefficient of electric amplitude transmission is, from relations (5)–(9),

$$t = E_t/E_i = 2A_1/(A_1 + A_2). \tag{14}$$

A corresponding definition for the coefficient of magnetic amplitude transmission is

$$t' = H_t/H_i = (A_2/A_1)t = 2A_2/(A_1 + A_2). \tag{15}$$

In the special situations for which the usual definitions of P_i and P_r are unique and correct, power reflection and transmission coefficients may be defined as

$$R = P_r/P_i = \mathrm{Re}(E_r H_r^*)/\mathrm{Re}(E_i H_i^*) = |r|^2, \tag{16}$$

$$T=P_t/P_i=\mathrm{Re}(E_tH_t^*)/\mathrm{Re}(E_iH_i^*)$$
$$=|t|^2\,\mathrm{Re}(A_2)/\mathrm{Re}(A_1) \qquad (17)$$
$$=|t'|^2\,\mathrm{Re}(A_2^{-1})/\mathrm{Re}(A_1^{-1}).$$

Note that in Eq. (17), T is not equal to $|t|^2$, as has been implied,[3] except when $\mathrm{Re}(A_2)=\mathrm{Re}(A_1)$.

At first thought it may appear that a unique separation of incident and reflected powers may be possible if the incident wave strikes the interface at an oblique angle. Detailed consideration reveals, however, that this is not so; that, in fact, fundamental results of the foregoing analysis are still true, and that the only modifications required are ones of detail. For example, if the electric vector is normal to the plane of incidence, and θ_1 represents the angle of incidence, all the equations which appear above may be retained provided that the following transformations are made.

[3] See reference 1, p. 513.

$$\begin{aligned}
&P_1\rightarrow P_1'=P_1\cos\theta_1\\
&P_2\rightarrow P_2'=P_2\cos\theta_2\\
&A_1\rightarrow A_1'=A_1\cos\theta_1\\
&A_2\rightarrow A_2'=A_2\cos\theta_2\\
&\cos\theta_2=[1-(P_1/P_2)^2\sin^2\theta_1]^{\frac{1}{2}}.
\end{aligned} \qquad (18)$$

Similarly, if the magnetic vector is normal to the plane of incidence, and θ_1 represents the angle of incidence, all of the equations may be retained provided that the following transformations are made.

$$\begin{aligned}
&P_1\rightarrow P_1'=P_1\cos\theta_1\\
&P_2\rightarrow P_2'=P_2\cos\theta_2\\
&A_1\rightarrow A_1'=A_1/\cos\theta_1\\
&A_2\rightarrow A_2'=A_2/\cos\theta_2\\
&\cos\theta_2=[1-(P_1/P_2)^2\sin^2\theta_1]^{\frac{1}{2}}.
\end{aligned} \qquad (19)$$

I wish to acknowledge the benefit of helpful discussion provided by Dr. O. D. Sledge of this Laboratory, and Dr. E. U. Condon of the National Bureau of Standards.

Recreation with Vector Algebra

In the following example, the magnitudes of all of the vectors and of all of their Cartesian components are rational integers.

Let

$$\begin{aligned}
\mathbf{A}&=\ \mathbf{i}+2\mathbf{j}+2\mathbf{k},\\
\mathbf{B}&=2\mathbf{i}-2\mathbf{j}+\ \mathbf{k},\\
\mathbf{C}&=2\mathbf{i}+\ \mathbf{j}-2\mathbf{k}.
\end{aligned}$$

Then, **A**, **B**, and **C**, are mutually perpendicular, so that

$$\mathbf{A}\times\mathbf{B}=3\mathbf{C} \qquad \mathbf{B}\times\mathbf{C}=3\mathbf{A} \qquad \mathbf{C}\times\mathbf{A}=3\mathbf{B}.$$

Now, let

$$\begin{aligned}
\mathbf{D}&=4\mathbf{B}+3\mathbf{C}=\ 14\mathbf{i}-\ 5\mathbf{j}-\ 2\mathbf{k},\\
\mathbf{E}&=4\mathbf{B}-3\mathbf{C}=\ \ 2\mathbf{i}-11\mathbf{j}+10\mathbf{k},\\
\mathbf{F}&=3\mathbf{B}+4\mathbf{C}=\ 14\mathbf{i}-\ 2\mathbf{j}-\ 5\mathbf{k},\\
\mathbf{G}&=3\mathbf{B}-4\mathbf{C}=-2\mathbf{i}-10\mathbf{j}+11\mathbf{k}.
\end{aligned}$$

Then, **D**, **E**, **F**, and **G**, are all perpendicular to **A**; also, **D** is perpendicular to **G**, and **E** is perpendicular to **F**, so that

$$\begin{aligned}
\mathbf{A}\times\mathbf{D}&=-3\mathbf{G} \qquad \mathbf{A}\times\mathbf{E}=3\mathbf{F}\\
\mathbf{A}\times\mathbf{F}&=-3\mathbf{E} \qquad \mathbf{A}\times\mathbf{G}=3\mathbf{D}\\
\mathbf{E}\times\mathbf{F}&=\mathbf{G}\times\mathbf{D}=75\mathbf{A}\\
\mathbf{E}\times\mathbf{D}&=\mathbf{G}\times\mathbf{F}=72\mathbf{A}\\
\mathbf{F}\times\mathbf{D}&=\mathbf{G}\times\mathbf{E}=21\mathbf{A}\\
-\mathbf{B}\times\mathbf{F}&=\mathbf{B}\times\mathbf{G}=\mathbf{C}\times\mathbf{D}=\mathbf{C}\times\mathbf{E}=-12\mathbf{A}\\
-\mathbf{B}\times\mathbf{E}&=\mathbf{B}\times\mathbf{D}=\mathbf{C}\times\mathbf{F}=\mathbf{C}\times\mathbf{G}=\ \ 9\mathbf{A}.
\end{aligned}$$

Amplifications and applications can easily be devised by those interested.

—Ralph Hoyt Bacon.

Concerning Rope Problems and the Principles of Momentum and Work

GORDON FERRIE HULL
Dartmouth College, Hanover, New Hampshire

THE older texts on mechanics contained problems like these: (i) A uniform, flexible rope is drawn from a heaped-up coil with constant speed along a frictionless floor; find the force;[1] or (ii) a coil of rope rests on the edge of an elevated table, a small portion of the rope being allowed to hang over the edge; find the subsequent motion.

Let us consider the first problem. (The second leads to a differential equation with an elliptic integral solution.) One is tempted to apply the principle of work, which means that the change in the kinetic energy of a body is equal to the work done on or by that body, but it can be shown that this principle can never be applied to a mass in motion which captures another mass, or whenever one mass is added to or subtracted from another. There are many ordinary illustrations: a rifle bullet entering a suspended block of wood; a locomotive moving under its own momentum and scooping up water out of a trough; a proton entering a nucleus. In all such cases there is a degradation or loss of mass kinetic energy, but the principle of momentum always applies. Let λ be the linear density of the rope and x the length of the portion of the rope in motion; then $F=\mathrm{d}(mv)/\mathrm{d}t$. Since v is constant, $F=v\mathrm{d}m/\mathrm{d}t=v\lambda\mathrm{d}x/\mathrm{d}t=\lambda v^2$. Substituting the numerical values given by Professor Sleator—$\lambda=2$ lb/ft, $v=5$ ft/sec—we obtain $F=50$ pdl.

But though the solution appears extremely simple, in reality we have ignored some disturbing features. There is a large gap between the ideal and actual. For not only is the rope flexible, it is also inextensible, and we would have some difficulty in coiling a rope so that a minute part of it might be put in motion without disturbing adjacent portions. If, for example, the rope were coiled round a frictionless drum, the force after the first millimeter of motion would be zero, since all the rope would have the same speed.

A closer approach to reality might be obtained in this problem. The rope AB is laid out in a straight line on a horizontal floor. A man draws the end A straight back on itself towards B with a speed of 5 ft/sec. Find the force. Here, since the rope is perfectly flexible, it is reasonable to say that a length $\mathrm{d}x$ can be bent back without setting in motion an adjacent part. Let a length x be drawn back; then as before $F=\mathrm{d}(mv)/\mathrm{d}t=v\mathrm{d}m/\mathrm{d}t=v\lambda\mathrm{d}x/\mathrm{d}t$. But now $\mathrm{d}x/\mathrm{d}t=v/2$, since the rope is bent back on itself. The man must walk 10 ft in order to set in motion 5 ft of rope. Hence, for the data given above, $F=\lambda v^2/2=25$ pdl.

There are some excellent problems in Wolstenholme's *Mathematical Problems*,[2] in which both the principles of work and of momentum are used in appropriate parts and in which a particle can be set in motion without disturbing the remaining portion of the body. There are 2814 problems in this collection,

> "Tricks to show the stretch of human brain,
> Mere curious pleasure or ingenious pain."[3]

For example, Problem 2354, page 419, is: "A large number of equal particles are attached at equal intervals to a string and the whole is heaped up close to the edge of a smooth horizontal table with the extreme particle just over the edge; prove that if v_n denotes the speed just before the $(n+1)$th particle is set in motion,

$$v_n^2=\frac{ag}{3}\frac{(n+1)(2n+1)}{n},$$

where a denotes the interval between two consecutive particles. Prove that when a is indefinitely diminished, the end of the string, in the limit, descends with uniform acceleration $g/3$. The whole energy dissipated just before the $(n+1)$th particle is set in motion is $amg(n^2-1)/6n$, where m is the mass of each particle."

Let u_n be the speed of the first n particles just after the nth particle has been set in motion; and v_n the speed after the nth particle has dropped a distance a just before the

[1] This problem was discussed by Professor Sleator, *Am. J. Physics* **15**, 474 (1947).

[2] J. Wolstenholme, *Mathematical Problems* (Mac Millan, London, 1878).

[3] A. Pope, *Essay on Man* (1733).

$(n+1)$th particle has been set in motion. Then

$$\tfrac{1}{2}nmv_n^2=\tfrac{1}{2}nmu_n^2+nmga,$$

or

$$v_n^2=u_n^2+2ga,$$

Then $(n+1)mu_{n+1}=nmv_n$, or

$$(n+1)^2u_{n+1}^2=n^2u_n^2+2n^2ga.$$

Form a series down to the relation $2^2u_2^2=u_1^2+2ga$. Since $u_1=0$, we obtain

$$(n+1)^2u_{n+1}^2=\frac{n(n+1)(2n+1)}{3}ga.$$

Then

$$u_{n+1}^2=\frac{n(2n+1)}{3(n+1)}ga$$

and

$$v_{n+1}^2=u_{n+1}^2+2ga=\frac{(n+2)(2n+3)}{3(n+1)}ga.$$

Changing $n+1$ to n gives

$$v_n^2=\frac{(n+1)(2n+1)}{3n}ga.$$

The value of the constant acceleration g', when the v's are large, is given by

$$g'=\frac{v_{n+1}-v_n}{t}=\frac{v_{n+1}^2-v_n^2}{2tv_n}=\frac{v_{n+1}^2-v_n^2}{2a}=\tfrac{1}{3}g.$$

Energy is dissipated here because of the sudden setting in motion of a new mass; there is no dissipation of energy during the constant acceleration. The energy lost is the difference between the work done $\sum_1^n namg$ and the kinetic energy $\tfrac{1}{2}nmv_n^2$, which equals $\frac{1}{6}\frac{n^2-1}{n}amg$.

It has been stated that the principle of work does not hold if mass is added to or subtracted from another mass. But the mass of a body increases owing to increased speed. Does the principle of work apply? It does; and so does the principle of momentum. We can show that $dE_k=Fdx$ and $dp=Fdt$ when E_k and p are the kinetic energy and momentum of a particle. For, with $v/c=\beta$,

$$dE_k/dx=d[m_0c^2/(1-\beta^2)^{\frac{1}{2}}]/dx$$

and

$$dp/dt=d[m_0c^2v/(1-\beta^2)^{\frac{1}{2}}]/dt$$
$$=d[m_0c^3\beta/(1-\beta^2)^{\frac{1}{2}}]/dt.$$

Here it is convenient to make

$$\frac{dp}{dt}=\frac{dp}{d\beta}\frac{d\beta}{dx}\frac{dx}{dt}=v\frac{dp}{d\beta}\frac{d\beta}{dx}$$

and

$$\frac{dE_k}{dx}=\frac{dE_k}{d\beta}\frac{d\beta}{dx}.$$

But we find that $dE_k/d\beta=vdp/d\beta$, and therefore that $dE_k/dx=dp/dt$, or $dE_k=vdp=vd(mv)$.

Incidentally, we may deduce the relation $E_m=mc^2$, where E_m is the total energy of mass m. Let us assume that the relations of classical mechanics hold, so that $dE/dx=dp/dt=vdp/dx$, and that the relation $m=m_0/(1-\beta^2)^{\frac{1}{2}}$ has been experimentally established. Then $m^2c^2-m^2v^2=m_0^2c^2$, or $c^2dm-v^2dm-mvdv=0$, or $c^2dm=vd(mv)$. From these relations, $dE_k=c^2dm$ and $E_k=c^2(m-m_0)$. Hence $E_m=mc^2$.

AMERICAN JOURNAL OF PHYSICS VOLUME 16, NUMBER 8 NOVEMBER, 1948

NOTES AND DISCUSSION

Utilizing the Vernier Principle for Precise Readings of Slide Rule Settings

ROGER WICKENDEN
Miami University, Oxford, Ohio

CONSIDERABLE space exists between adjacent division marks on a slide rule. When readings are being made, the hairline generally lies between two division marks, and the decimal part of a division is often incorrectly estimated. We shall attempt to show how by using the vernier principle one can make slide rule readings and settings with increased accuracy, perhaps with an additional significant figure for some types of readings. We assume that the slide rule is constructed with sufficient precision to warrant more accurate readings.

To use the slide rule as a vernier, one sets the right-hand index of the *C* scale directly over the 9 on the *D* scale; then for every ten divisions anywhere along the *C* scale there are nine on the *D* scale. Thus the *C* scale becomes a vernier to read the *D* scale.

The examples given below are solved on adjacent *C* and *D* scales of a 10-inch Mannheim-type slide rule. The principles involved may also be applied to the *A* and *B* scales, and to rules of other lengths.

(a) Suppose that, as the result of a computation, the hairline lies between the divisions 136 and 137 on the D scale and we wish to know what fraction of a division lies to the left of the hairline.

With the hairline already placed in this position, we set the C scale in the manner just shown so that it becomes a vernier to read the D scale. We now note the division mark on the C scale (the vernier) which lies closest to the hairline, in this example either 151 or 152. Next, the vernier must be shifted slightly so that this division mark lies directly under the hairline. Calling this division mark zero, one must count to the right until a vernier and a scale division mark coincide. The number of this vernier division mark will be the decimal part of the scale division that we wished to find.

(b) Suppose we wish to divide 288 by 645. Upon making very careful settings in the traditional manner, we obtain a quotient of around 0.446 or 0.447. Holding this quotient with the hairline, we can move the C scale so that its right-hand index lies directly over the 9 on the D scale, and the C scale becomes a vernier.

Shifting the C scale slightly to line up the nearest mark (in this case, 495) with the hairline, and counting to the right, the third division mark is seen to be lined up with a division mark on the D scale. Thus, to the left of the hairline is 3/10 of a division and our reading is 0.445 +3/10(0.005), or 0.4465. Although the last figure may often be in error, we still have a quotient of four significant figures, which is better than the usual three.

General theory. In a sense, the vernier has been used to "divide" each scale division into ten parts. There is no reason why we must limit ourselves to tenths of a division. The slide rule vernier can be set to divide each division into five, eight, twelve, twenty, or indeed, any convenient number of parts, according to which numbers on the C and D scales are originally lined up.

Let n be the number of parts into which we wish to divide each scale division. Then there must be n divisions on the vernier for $(n-1)$ divisions on the scale. Let r be the true reading of any point on the C scale and R that of the point directly underneath it on the D scale. For a given setting of the C scale, r/R will be constant all along the scales. Therefore, $(r+n)/(R+n-1)=r/R$. Subtracting the right-hand numerator from the left-hand numerator and the right-hand denominator from the left-hand denominator yields $n/(n-1)=r/R$. Thus, if we wish n to be 20, we set our rule so that r/R is equal to 20/19. In other words, we line up 20 on the C scale with 19 on the D scale. If n is to be 15, we line up 15 on the C scale with 14 on the D scale.

A conversion problem. Not only readings, but accurate settings can be made using the vernier principle. Suppose we are without tables and wish to convert a given number of degrees and minutes to radians. For instance, how many radians are there in 65°26′? If we let y equal the number of radians, then $y=(65+26/60)(\pi/180)$.

In performing this computation on the slide rule, we first wish to set the hairline on 65 plus 26/30 of a one-half unit division, which is the same as 65.5 minus 2/15 of a division. We set the hairline at the estimated position. We line up 15 on the C scale with 14 on the D scale, thus setting up our vernier to divide each scale division into 15 parts. Now we shift the C scale slightly so that the nearest division mark is moved under the hairline. We will call this the zero mark.

If the estimated position were exactly correct, the thirteenth division mark to the right or the second division mark to the left on the vernier would now coincide with a scale division. Since, in general, it does not coincide, we shift the vernier slightly to make it do so. Resetting the hairline to coincide with the zero mark on the vernier, we obtain the exact setting for 65°26′.

Proceeding with the multiplication and division, but using no more vernier settings, a result of slightly more than 1.14 radian is obtained. By utilizing the vernier principle, however, it can be read accurately as 1.142 radian, if the preceeding settings were made with sufficient care. This compares favorably with the true value of 1.14203, obtained by arithmetic computation.

It should be mentioned that there are places along the scale where the illustrated procedure will not apply without modification. For example, when the right-hand index of the C scale is lined up with the 9 on the D scale, the ratio of ten vernier divisions to nine scale divisions does not hold at the three places where the C and D scales are divided into different kinds of divisions. These places on the D scale are between 1.8 and 2, between 3.6 and 4, and between 9 and the right-hand index. In order to apply the vernier principle at these three places, the C scale may be reset to give the desired ratio, or a different ratio may be employed. For instance, when a ratio of ten to nine divisions is desired, a ratio of five to nine divisions can often be used satisfactorily.

The writer is deeply grateful to Dr. R. L. Edwards for his valuable suggestions and interest in the writing of this paper.

Simple Apparatus for Demonstrating A.C.-D.C. Voltage Relationships

LEO SEREN*
University of Idaho, Moscow, Idaho

THE simple apparatus represented in Fig. 1 has been found useful in demonstrating the relationship between root mean square a.c. voltage and average, maximum and instantaneous d.c. voltages. The circuit, including a 6X5 rectifier, is self-explanatory. Four basic experiments are listed below.

1. *Maximum d.c. voltage and root-mean-square a.c. voltage.* With the switch open and the condenser connected to BB, a d.c. vacuum-tube voltmeter at BB reads maximum d.c. voltage which can be compared with the reading of an a.c. voltmeter at AA. Experimentally this ratio comes out quite close to $2^{\frac{1}{2}}$, the theoretical ratio for an emf varying according to a sine curve.

2. *Full wave and half wave root-mean-square voltages.* With the condenser removed, the a.c. voltmeter is connected to BB. The full wave voltage is read with the switch closed

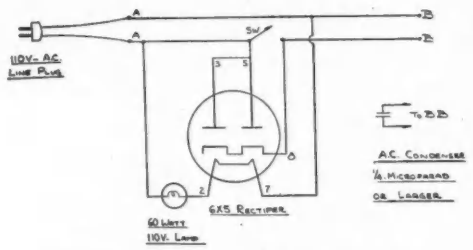


FIG. 1. Simple apparatus for demonstrating a.c.-d.c. voltage relationships.

and the half wave "voltage" is obtained with the switch open. Experimentally this ratio also comes out quite close to $2^{\frac{1}{2}}$. The fact that the ratio is *not* 2 when half of the voltage wave is removed is a direct experimental proof that the a.c. voltmeter does not measure a linear average. The $2^{\frac{1}{2}}$ corresponds to the ratio of the root mean square of the full and the half sine wave.

3. *D.c. maximum and d.c. average of half sine wave voltage.* With the switch open, a d.c. vacuum-tube voltmeter is connected to *BB*, with and then without the condenser. The reading with the condenser is the maximum of the sine wave voltage; the reading without the condenser is the linear average of the half sine wave. Experimentally this ratio comes out close to π, the theoretical ratio for a pure sine wave input. The experimental ratio is not as close as in cases 1 and 2, probably because the d.c. voltmeter exhibits some inductance for the half-wave alternating current.

4. It is pedagogically useful to connect the d.c. voltmeter to *BB* with the switch closed. The reader can predict the results or do the experiment.

* Now at Reed College, Portland, Oregon.

Maxwell's Thermodynamic Relations

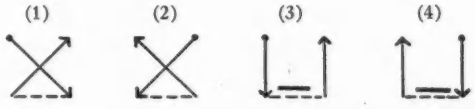
CHARLES M. FOCKEN
University of Otago, Dunedin, New Zealand

IN thermodynamic theory the four partial derivative equations usually called Maxwell's thermodynamic relations are frequently used. They may be written in an arbitrary order, as follows:

$$(\partial P/\partial T)_V = (\partial S/\partial V)_T, \tag{1}$$
$$(\partial V/\partial S)_P = (\partial T/\partial P)_S, \tag{2}$$
$$(\partial P/\partial S)_V = -(\partial T/\partial V)_S, \tag{3}$$
$$(\partial V/\partial T)_P = -(\partial S/\partial P)_T. \tag{4}$$

I do not agree entirely with Zemansky's statement:[1] "It is not necessary for the student to memorize the equations since they are so easily derived." The derivation of the four relations is laborious and not especially easy. Furthermore, one particular equation is often required, in which case it is far from easy to quote the particular one. In fact I have experienced difficulty in writing down from memory (without proof) the four relations, unless I happened to have been using them not more than a few weeks previously. Probably physicists who are continually using thermodynamics have no difficulty at all, but there may be many university teachers like myself who, because of other teaching assignments, have some trouble remembering these useful equations. If there is any simple and reliable aid for remembering the equations in detail it has escaped my notice.

A mnemonic that I concocted suffered from the one unpardonable fault of a mnemonic; it occasionally eluded the memory. In a course on thermodynamics which I give to third year science students I have, for several years, referred to this difficulty. At the appropriate time I have asked any student who has read about or devised a good aid to the memory to tell me about it. I am indebted to Mr. E. I. White for the following mnemonic which, although far from perfect, is the best I have encountered. We first write the array $\begin{vmatrix} P & V \\ S & T \end{vmatrix}$ consisting of the two associated pairs P, V and S, T each arranged in alphabetical order. Then the four relations can be written down directly from this array if the quantities are selected in the order indicated by the symbolic signs



For example, from the first sign we write Eq. (1) $(\partial P/\partial T)_V = (\partial S/\partial V)_T$, adding the appropriate suffixes, and from the fourth sign we write Eq. (4) $(\partial V/\partial T)_P = -(\partial S/\partial P)_T$, adding the minus sign, as well as the suffixes.

Such standard texts as I have consulted provide little help in the difficulty of remembering the relations. Saha and Srivastava, however, use a method[2] which enables the relations to be written out. It can be shown that $\partial(P, V)/\partial(x, y) = \partial(T, S)/\partial(x, y)$, where each of these expressions stands for a Jacobian. For example, the first is

$$\begin{vmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} \end{vmatrix}.$$

Any two of the quantities P, V, T, S can be chosen as the independent variables x, y. This can be done in six different ways, which lead after some simple manipulation to the four Maxwell relations and two less important equations. This is straightforward to remember, but it is lengthy as well as rather tricky to use in order to obtain one or two of the relations. For this reason I much prefer, as a mnemonic only, the one devised by Mr. White. If any other reader has a better one I should like to know it.

[1] M. W. Zemansky, *Heat and thermodynamics* (McGraw-Hill, 1943), p. 221.
[2] Saha and Srivastava, *A treatise on heat* (Indian Press, 1931), p. 463.